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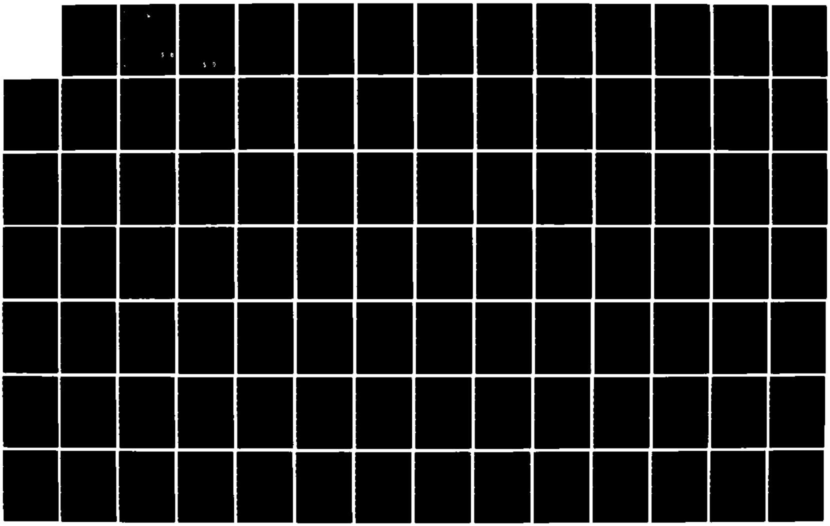

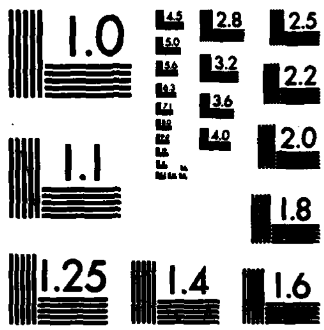

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SCHOTTKY THEORY OF THREE

COMPONENT PLASMAS

THESIS

Christopher J. Clouse
Second Lieutenant, USAF

AFIT/GNE/~~EN~~ ENP/85M-3

DEPARTMENT OF THE AIR FORCE
AIR UNIVERSITY
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SCHOTTKY THEORY OF THREE-COMPONENT PLASMAS

THESIS

Presented to the Faculty of the School of Engineering
of the Air Force Institute of Technology
Air University
In Partial Fulfillment of the
Requirements for the Degree of
Master of Science in Nuclear Engineering

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Second Lieutenant, USAF

January 1985

Approved for public release; distribution unlimited

Acknowledgements

Many thanks to my advisor, Lt. Col. William Bailey, for his invaluable assistance and to Dr. David Lee for his readily available help concerning problems encountered while reviewing his paper.

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List of Symbols

α -ratio of normalized negative ion to electron densities (n_-/n_e)

β -ionization rate

D_+, D_-, D_e -diffusion coefficients for positve ions, negative ions and electrons respectively

D_+^a, D_-^a, D_e^a -ambipolar diffusion coefficients for positive ions, negative ions and electrons respectively

E -electric field

ϵ_2 -distance from outer wall where Lee's integration routine is forced to match his Taylor series expansions

ϕ -associative detachment rate

γ - Θ_e/Θ_g

$\Gamma_+, \Gamma_-, \Gamma_e$ -fluxes of positive ions, negative ions and electrons respectively

h - n_{-o}/n_{eo}

λ -dissociative attachment rate

m_s -mass of species s

μ_+, μ_-, μ_e -mobilities of positive ions, negative ions and electrons respectively

n_+, n_-, n_e -charged particle densities of positive ions, negative ions and electrons respectively

n_{+o}, n_{-o}, n_{eo} -charged particle densities on axis

ν -ionization rate

ν_+, ν_-, ν_e -collision frequencies of positive ions, negative ions and electrons respectively

T_+, T_-, T_e -temperatures of the postive ions, negative ions and electrons respectively

Θ_e, Θ_g -electron and backround gas temperature in eV

X, Y -normalized charged particle profiles $\frac{n_e}{n_{eo}}$ and $\frac{n_-}{n_{-o}}$

Y_1, Y_2 -given by eq.C-3

Y_3, Y_4

z -normalized coordinate

ABSTRACT

The purpose of this study was to analyze and compare papers by Lee (ref. 7), Thompson (refs. 9 and 10) and Ingold (ref. 6) which give conflicting results concerning the charged particle profiles in an oxygen discharge tube. Lee predicts proportional negative ion and electron profiles whereas Thompson and Ingold predict non-proportional profiles.

The analytic developments were critically reviewed and numerical solutions were developed for each approach. It was found that Thompson's results could not be obtained without introducing the additional assumption:

$$\frac{\nabla n_-}{\nabla n_e} \approx \gamma \alpha$$

Using this relation and the assumption that $\frac{n_e}{n_{eo}} \simeq 1$ (as is indicated by Thompson's experimental results) a new analytic relation was developed for the negative ion profile which agreed well with Thompson's experimental results.

Ingold's development, on the other hand, seemed to be lacking a firm mathematical basis. One of Ingold's fundamental relations used throughout his development could not be established and, furthermore, it was shown that a completely different solution could be developed without introducing further constraints. Also, the numerical solution developed for Ingold did not give profiles similar to Ingold's.

Lee's profiles reduce to Thompson's when associative detachment is ignored and a common set of parameters and coordinate system are used. When associative detachment is included, Lee and Thompson do not give similar profiles. Thus, the relation

$$\frac{\nabla n_-}{\nabla n_e} = \gamma \alpha$$

may only be valid in limiting cases (e.g. when associative detachment can be ignored.) Results from an investigation of power series solutions indicated that they could probably be used to obtain Lee's profiles, thus negating the need to numerically integrate simultaneous first order equations, as is done by Lee.

I. INTRODUCTION

The low pressure gas discharge with two charged species, positive ions and electrons, was modeled by Schottky in 1924. Predictions of his theory have agreed well with experimental results. Recently, though, attention has been focused on the behavior of electronegative gas discharges because of their applicability to semiconductor processing, lasers and ion sources. To model electronegative gases, Schottky theory must be extended to include negative ions. Several attempts have been made to incorporate negative ions but results have not been in agreement. Lee claims that the charged particle profiles are proportional to one another but Thompson and Ingold claim the profiles are not proportional.

Problem

This study will critically analyze current papers on three component diffusion, outline assumptions, and develop numerical solutions in order to compare and contrast the various approaches and possibly draw conclusions concerning their validity.

The study will be limited to papers by Thompson (Ref 9 and 10), Lee (Ref 7), and Ingold (Ref 6).

Assumptions

All papers studied begin with the diffusion equations

$$\Gamma_+ = -D_+ \nabla n_+ + \mu_+ n_+ E \qquad \text{(1-1a)}$$

$$\Gamma_- = -D_- \nabla n_- - \mu_- n_- E \tag{1-1b}$$

$$\Gamma_e = -D_e \nabla n_e - \mu_e n_e E \tag{1-1c}$$

where Γ is the particle flux density and D is the diffusion coefficient which can be defined through the Einstein relation as

$$D_s = \frac{kT_s}{q} \mu_s \tag{1-2}$$

μ is the mobility defined as (4:186)

$$\mu_s = \frac{q_s}{m_s \nu_{sN}} \tag{1-3}$$

where ν_{sN} is the collisional frequency between the species and the neutral particles in the plasma. n represents the species number density and E is the electric field. Thus, it is assumed that the diffusion equations are applicable. The assumptions necessary in obtaining the diffusion equations from first principles are: no external forces acting on the charged species other than an electric field, neutral particles have no mean velocity, pressure is hydrostatic, and the drift velocity of the charged species is assumed to be negligible compared to the mean thermal velocity.

Approach and Presentation

Research began with a thorough analysis and critique of each paper studied. This was necessary before any further work could

be initiated and offered some insight as to how much further work each paper warranted. Next, numerical solutions yielding particle profiles were developed for each approach. This facilitated comparison and contrast especially when changes and variations were made to each theory. Next, an unsuccessful attempt was made to directly integrate the three diffusion equations, three continuity equations and Poisson's equation (seven equations and seven unknowns). Finally, a more detailed analysis was made of the power series expansions (both on axis and at the wall) used by Lee to investigate the possiblities of using power series expansions to obtain profiles over the entire tube radius, thus eliminating any need for direct integration of equations.

This presentation will begin with the assumptions made by each paper, followed by discussions of: the derivation of the equations found in each paper (many of these derivations can be found in the appendices), the developments of the numerical solutions, the behavior of the equations in the two component limit and the sensitivity of each solution to a variation of parameters. The presentation concludes with a comparison of results and a short discussion of boundary conditions.

<u>Backround</u>

As important backround information, we will now briefly redevelop the two-component Schottky solution (4:187).

We assume

$$n_+ = n_e = n$$
$$\Gamma_e = \Gamma_+ = \Gamma \qquad (1-4)$$

where n stands for charged particle density and Γ is particle current density or flux. 1-1 can then be written as

$$\Gamma = -D_e \frac{dn}{dz} - n\mu_e E$$
$$\Gamma = -D_+ \frac{dn}{dz} + n\mu_+ E \qquad (1-5)$$

where the gradients are in Cartesian coordinates. Eliminating E and solving for Γ , we have

$$\Gamma = -D_a \frac{dn}{dz} \qquad (1-6)$$

where

$$D_a \equiv \frac{\mu_+ D_e + \mu_- D_+}{\mu_+ + \mu_-} = \frac{\Theta_e + \Theta_q}{\frac{1}{\mu_+} + \frac{1}{\mu_e}} \qquad (1-7)$$

This last relation was obtained by employing the Einstein relation

$$\frac{D_s}{\mu_s} = \frac{kT_s}{q} \equiv \frac{\Theta_s}{q} \qquad (1-8)$$

Using the continuity equation

$$\frac{d\Gamma}{dz} = \nu n \qquad (1\text{-}9)$$

in 1-6, we obtain

$$-D_a \frac{d^2 n}{dz^2} = \nu n \qquad (1\text{-}10)$$

which yields the solution

$$n = n_o \cos \sqrt{\frac{\nu}{D_a}}\, z \qquad (1\text{-}11)$$

Since we require the number density to go to zero at the wall, we must require the relation

$$\sqrt{\frac{\nu}{D_a}} = \frac{\pi}{2} \qquad (1\text{-}12)$$

to be true.

II. ASSUMPTIONS AND ANALYTIC DEVELOPMENTS

Besides the assumptions implicit in the diffusion equations, certain other assumptions are made. Three of these are common to all three papers analyzed. The first of these is quasineutrality i.e.

$$n_+ \simeq n_- + n_e \tag{2-1}$$

The second is that the net flow of charge is zero everywhere i.e.

$$\Gamma_+ = \Gamma_- + \Gamma_e \tag{2-2}$$

The third is that both the negative ion and positve ion temperatures are equal to the backround gas temperature.

Lee further assumes that there are three production and loss terms:

ionization= $\nu(T_e)n_e \mathrm{cm}^{-3}\ \mathrm{sec}^{-1}$ (2-3a)

dissociative attachment= $\lambda(T_e)n_e\ \mathrm{cm}^{-3}\ \mathrm{sec}^{-1}$ (2-3b)

associative detachment= $\varphi(T_e)n_e\ \mathrm{cm}^{-3}\ \mathrm{sec}^{-1}$ (2-3c)

Lee also assumes that the charged particle densities are zero at the outer boundary and, for his analytic solution, he requires the following proportionality relation to be true:

$$\frac{n_-}{n_{-o}} = \frac{n_e}{n_{eo}} \tag{2-4}$$

Ingold uses ionization and dissociative attachment as production and loss terms but ignores associative detachment. Also, like Lee, Ingold assumes the particle densities go to zero at the outer boundary.

Thompson uses the same production and loss terms as Ingold. Furthermore, although Thompson doesn't specifically state the assumption of a proportionality relation, it was determined that the relation

$$\frac{\nabla n_-}{\nabla n_e} = \gamma \alpha \qquad (2\text{-}5)$$

is needed to obtain Thompson's results. (See appendix A)

<u>Discussion of Derivation of Equations</u>

Like Edgley and Von Engel (2), Lee initially develops a system of seven equations and seven unknowns. These seven equations are the three diffusion equations, the three continuity equations (see eqs. B-1 and B-2) and Poisson's equation

$$\nabla \cdot E = \frac{q}{\epsilon_0}(n_+ - n_- - n_e) \qquad (2\text{-}6)$$

These are reduced to a system of four equations and four unknowns by imposing 2-1 and 2-2 and eliminating the electric field from the set of equations. In general, Lee's step by step development is detailed enough such that analysis of his work consisted mainly of checking for algebraic mistakes -- none of which were found in his analytic development. Thus, no

appendix was necessary for tracing the derivation of Lee's equations.

Ingold, like Lee, eliminates the electric field from his equations. His development, however, seems to be seriously flawed in that the relation (eq. B-26)

$$x'(ax+by) = y'(cx+dy) \tag{2-7}$$

which is essential to his derivation, could not be established. (See appendix B.) Rather, the relation (eq. B-30)

$$x''(ax'+by') = y''(cx'+dy') \tag{2-8}$$

where a,b,c and d are constants, seems to be the correct equation.

It is possible to re-develop Ingold's equations using 2-8. The derivation is as follows.

Dividing 2-8 through by x' we have

$$x''\left(a+b\frac{y'}{x'}\right) = y''\left(c+d\frac{y'}{x'}\right) \tag{2-9}$$

As the spatial coordinate, z, goes to zero, the ratio y'/x' becomes indeterminate since both derivatives go to zero. Using L'hopital's rule we obtain

$$\lim_{z\to 0}\frac{y'}{x'} = \frac{y_o''}{x_o''} \tag{2-10}$$

where the subscript indicates the variable is evaluated at z=0. Thus 2-9 becomes

$$X_0''\left(a+b\frac{Y_0''}{X_0''}\right)=Y_0''\left(c+d\frac{Y_0''}{X_0''}\right) \tag{2-11}$$

Multiplying through by X_0'' and rearranging, we have

$$aX_0''^2+(b-c)X_0''Y_0''-dY_0''^2=0 \tag{2-12}$$

Dividing through by $Y_0''^2$, we obtain a quadratic equation with the solution

$$\frac{X_0''}{Y_0''}=\frac{c-b\pm\sqrt{(b-c)^2+4ad}}{2a}\equiv\rho \tag{2-13}$$

This relation replaces B-29 and must be used in it's place when substituted into B-24. Thus B-24 becomes

$$\Theta_e X_0''-\frac{\nu-\lambda}{\mu_e}\left[\frac{(\Theta_e+\Theta_g)X_0''+2\Theta_g\frac{n_{-0}}{n_{e0}}\frac{X_0''}{\rho}}{\frac{\nu-\lambda}{\mu_e}+\frac{\nu}{\mu_+}+\frac{\lambda}{\mu_-}}\right]$$

$$=\Theta_g Y_0''-\frac{\lambda}{\mu_-}\left[\frac{(\Theta_e+\Theta_g)X_0''+2\Theta_g\frac{n_{-0}}{n_{e0}}\frac{X_0''}{\rho}}{\frac{\nu-\lambda}{\mu_e}+\frac{\nu}{\mu_+}+\frac{\lambda}{\mu_-}}\right]\frac{n_{e0}}{n_{-0}} \tag{2-14}$$

which simplifies to

$$\frac{\nu-\lambda}{\mu_e}=\frac{\left(\frac{\nu-\lambda}{\mu_e}+\frac{\nu}{\mu_+}+\frac{\lambda}{\mu_-}\right)\left(\Theta_e-\Theta_g\frac{1}{\rho}\right)}{\Theta_e+\Theta_g+2\Theta_g\frac{n_{-0}}{n_{e0}}\frac{1}{\rho}}+\frac{\lambda n_{e0}}{\mu_- n_{-0}} \tag{2-15}$$

which differs considerably from B-31.

The next change necessary is to replace B-44 with

$$y' = u + f x' \tag{2-16}$$

With this change, we can proceed exactly as Ingold does to obtain the parametric solutions

$$x' = A' u^s + B u \tag{2-17a}$$

$$y' = C' u^s + D u \tag{2-17b}$$

as opposed to B-50 and B-52. The difference between A and A' and C and C' is that A and C can be determined by the boundary conditions

$$x_0 = 1 \text{ and } y_0 = 1 \tag{2-18}$$

whereas A' and C' cannot be determined by the parallel boundary condition

$$x_0' = 0 \text{ and } y_0' = 0 \tag{2-19}$$

Therefore, another boundary condition would be needed for further development of this solution. If B-54 is differentiated to give

$$x''' + \frac{2h}{\gamma+1} y''' + k^2 x' = 0 \tag{2-20}$$

then 2-17a and b can be substituted in to give an equation identical to B-56 except that A and C are repaced with A' and C'.

Even this solution seems somewhat arbitrary as will now be demonstrated.

It was pointed out in appendix B that eq. B-47 seemed to be an arbitrary assumption fixing the value of f. Suppose, we now fix the value of f such that

$$b - df = 0 \quad \text{or} \quad f = b/d \tag{2-21}$$

Then B-49 becomes

$$\frac{du}{dx'} = \frac{nx' + pu}{mx'} \tag{2-22}$$

where x is replaced by x' to conform with 2-16. Also, the constants take on new values such that

$$\begin{aligned} m &= a + bf - f(c + df) \\ n &= c + df \; ; \; p = d \end{aligned} \tag{2-23}$$

Rearranging 2-22, we have

$$\frac{dx'}{du} - \frac{p}{m}\frac{u}{x'} = \frac{n}{m} \tag{2-24}$$

The solution to the homogenous form of this equation is

$$X' = R\sqrt{\tfrac{p}{m}}\,u \tag{2-25}$$

where R is an arbitrary constant. Assume a particular solution of the form

$$X' = G\,u \tag{2-26}$$

Therefore

$$X' = \left(R\sqrt{\tfrac{p}{m}} + G\right)u \tag{2-27}$$

Substituting 2-27 into 2-24 and simplifying, we have

$$mG^2 + \left(2mR\sqrt{\tfrac{p}{m}} - n\right)G - nR\sqrt{\tfrac{p}{m}} + (R^2-1)p = 0 \tag{2-28}$$

Solving for G, we obtain

$$G = \frac{\left(n - 2mR\sqrt{\tfrac{p}{m}}\right) \pm \sqrt{4mR^2p - 4mnR\sqrt{\tfrac{p}{m}} + n^2 + 4m\left(nR\sqrt{\tfrac{p}{m}} + (R^2-1)p\right)}}{2m} \tag{2-29}$$

Substituting 2-27 into 2-16, we have

$$Y' = \left(\tfrac{b}{d}R\sqrt{\tfrac{p}{m}} + \tfrac{b}{d}G + 1\right)u \tag{2-30}$$

Making the following definitions

$$\Phi \equiv \frac{b}{d} R\sqrt{\frac{\rho}{m}} + \frac{b}{d} G + 1$$

$$\Psi \equiv R\sqrt{\frac{\rho}{m}} + G \qquad (2\text{-}31)$$

and substituting 2-30 and 2-27 into 2-20 we obtain, after simplification,

$$\left(\Psi + \frac{2R}{\gamma H}\Phi\right)u'' + k^2\Psi u = 0 \qquad (2\text{-}32)$$

The solution to this second order differential equation can be seen by inspection to be some combination of cosines and sines. This differs considerably from B-55 which has no obvious analytic solution.

Thompson, like Lee and Ingold, eliminates E from his equations. Furhtermore, an unusual proportionality relation

$$\frac{\nabla n_-}{\nabla n_e} = \gamma \alpha \qquad (2\text{-}33)$$

is needed to obtain Thompson's ambipolar diffusion coefficients (see appendix A). The fact that this relation is required can be further demonstrated by substituting A-2 into the assumption of zero total charge flux

$$\Gamma_+ = \Gamma_- + \Gamma_e \qquad (2\text{-}34)$$

giving

$$-D_+^a \nabla n_+ = -D_-^a \nabla n_- - D_e^a \nabla n_e \qquad (2\text{-}35)$$

Since $\nabla n_+ = \nabla n_- + \nabla n_e$, we can simplify 2-35 to

$$\frac{\nabla n_-}{\nabla n_e} = \frac{D_+^a - D_e^a}{D_-^a - D_+^a} \qquad (2\text{-}36)$$

From A-3, we can evaluate the right side of 2-36 to give

$$\frac{D_+^a - D_e^a}{D_-^a - D_+^a} = \frac{\dfrac{1+\alpha\mu_-/\mu_e}{1+\gamma\alpha} - 1}{\dfrac{1}{\gamma}\dfrac{\mu_-}{\mu_e} - \dfrac{1+\alpha\mu_-/\mu_e}{1+\alpha\gamma}}$$

$$= \frac{\alpha\gamma\left(\frac{1}{\gamma}\frac{\mu_-}{\mu_e} - 1\right)}{\frac{1}{\gamma}\frac{\mu_-}{\mu_e} - 1} = \alpha\gamma \qquad (2\text{-}38)$$

Therefore

$$\frac{\nabla n_-}{\nabla n_e} = \alpha\gamma \qquad (2\text{-}38)$$

This relation was required to obtain Thompson's expressions for D_e^a, D_-^a and D_+^a however D_e^a could also be obtained from the assumption that $\mu_- = \mu_+$.

Using Cartesian coordinates, as Thompson does, we can write 2-38 as

$$\frac{1}{n_-}\frac{dn_-}{dz} = \gamma \frac{1}{n_e}\frac{dn_e}{dz} \qquad (2\text{-}39)$$

Integrating, we have

$$\ln n_- - \ln n_{-o} = \gamma(\ln n_e - \ln n_{eo}) + C \qquad (2\text{-}40)$$

Taking the exponential of both sides we obtain

$$\frac{n_-}{n_{-o}} = C\left(\frac{n_e}{n_{eo}}\right)^{\gamma} \qquad (2\text{-}41)$$

where C=1 since

$$1 = C(1)^{\gamma} \text{ at } z=0 \qquad (2\text{-}42)$$

An interesting relation can be developed if we assume $\frac{n_e}{n_{ao}} \simeq 1$ as is indicated by Thompson's experimental results throughout most of the discharge tube. Taking the power series expansions for ln(1+x), we have

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} \cdots \qquad (2\text{-}43)$$

If we define x as

$$x = \frac{n_e}{n_{eo}} - 1 \qquad (2\text{-}44)$$

then x<<1 and we can write

$$\ln\left(1+\frac{n_e}{n_{eo}}-1\right)=\ln\frac{n_e}{n_{eo}}\simeq\frac{n_e}{n_{eo}}-1 \qquad (2\text{-}45)$$

So 2-40 becomes

$$\ln\frac{n_-}{n_{-o}}=\gamma\left(\frac{n_e}{n_{eo}}-1\right)$$

or

$$n_-=n_{-o}\exp\left[\left(\frac{n_e}{n_{eo}}-1\right)V_e/V_-\right] \qquad (2\text{-}46)$$

where V_e =electron temperature and V_- =negative ion temperature. This resembles the Boltzmann distribution given by Thompson (10:820) to model the negative ion concentration as given by his experimental results. The exact form given by Thompson is

$$n_-\doteq n_{-o}\exp\left[V/V_-\right] \qquad (2\text{-}47)$$

where V is the spatial potential relative to the discharge axis (z=0). On page 819 of ref. 10, Thompson plots V/V_e according to experimental measurements.

From Thompson's graph of n_e/n_{eo} (10:820) one can see, without too much imagination, that a graph of ($\frac{n_e}{n_{eo}}$ - 1) would look very similar to a graph of V. It is difficult to say

how closely they match since Thompson doesn't give any specific numbers and his graphs obviously weren't drawn for the purpose of reading off specific data points. Nonetheless, it is interesting to note that an analytic expression can be obtained from the assumptions

$$\frac{\nabla n_-}{\nabla n_e} = \gamma\alpha \quad \text{and} \quad \frac{n_e}{n_{e0}} \simeq 1$$

which seems to agree fairly well with experimental results given by Thompson.

III. NUMERICAL SOLUTIONS AND RESULTS

Discussion of numerical solutions

Because Lee uses cylindrical coordinates, he is forced to deal with singularities at the wall and on axis. The singularity on axis is dealt with by using the power series expansions given by C-6. The coefficients are found through the recursion relations C-11, C-14, C-16, and C-23 and then a small value for z is chosen so initial values for y_1 , y_2 , y_3 and y_4 can be obtained for use in the integration of equations C-1a,b,c and d. Fig. 1 depicts the normalized charged particle profiles, given by the code outlined in appendix D, as functions of the normalized radial coordinate, z.

The electric field, as evaluated in the code, is shown in fig. 2. The field is seen to rise sharply as one leaves the axis, pass through an inflection point at about z=0.5 and rise asymptotically as the outer wall is approached. This latter behavior is to be expected since we have required the charged particle densities to go to zero at the wall.

The singularity at the outer wall is dealt with by the formal approximations given by C-24. It is shown in appendix C that C-24 reduces to C-25 when terms of order $(1-z)^2$ and larger are neglected. The integration routine proceeds to a point, z, a small distance , ϵ_2 , away from the outer wall. The values of y_1 and y_2 at this point are used to find a_1 and b_1 in C-25. Given a_1 and b_1 , we can find c_0 and d_0 as

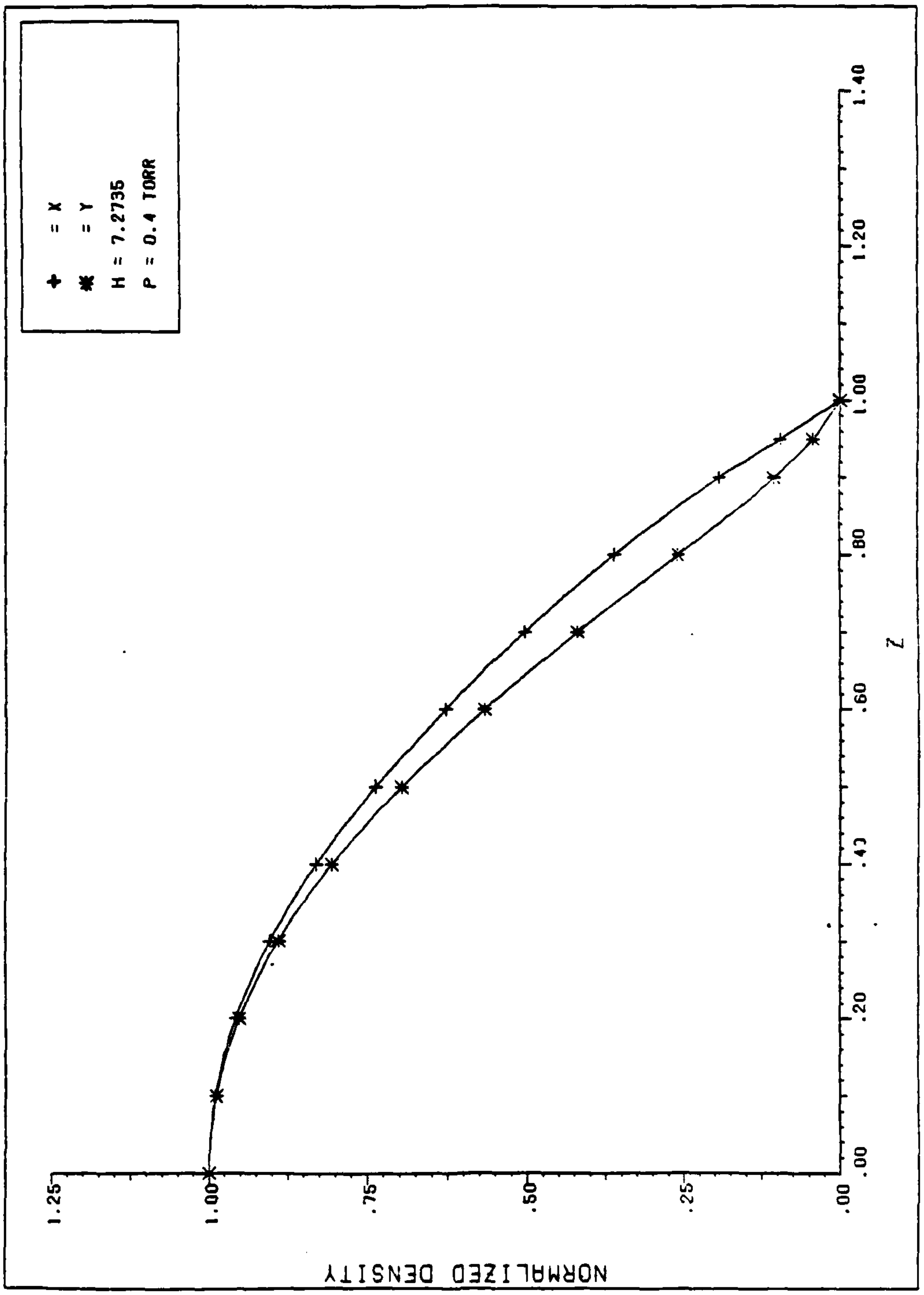

FIG. 1 CHARGED PARTICLE PROFILES (LEE)

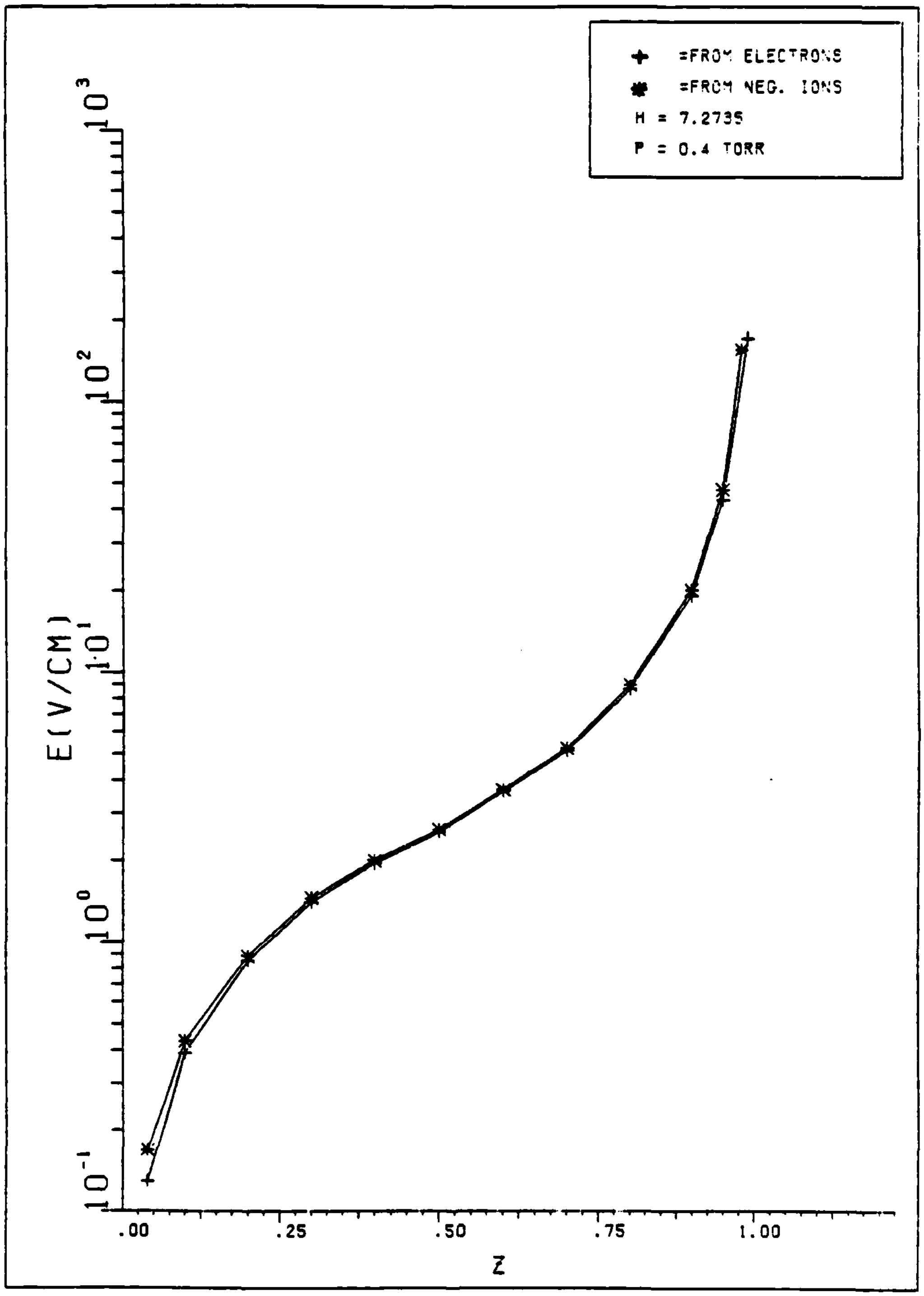

FIG. 2 ELECTRIC FIELD PROFILES (LEE)

defined in C-31 and C-33. Given C_0 and d_0, we are now able to evaluate y_3 and y_4 as given by C-25 (at $1-\epsilon_2$) and compare these values with those obtained from the integration routine at the point $1-\epsilon_2$. If the two sets of values differ, then the adjustable parameters -- pressure, electron temperature, and h -- must be varied. The profile in fig. 1 was obtained by holding the pressure constant at 0.4 torr and varying h until a minimum was found in the differences between the values of y_3 and y_4 obtained from the integration routine and the Taylor series expansion near the wall. The electron temperature was then varied and the process was repeated. This was done a number of times to find the best electron temperature for the given pressure. Further discussion of the sensitivity of the profiles to a variation of the parameters will be made at the end of the chapter.

Finally, although Lee intended his power series approximations on axis to be used only in the near vicinity of the axis, it was found that the series gave profiles which agreed remarkably well with those obtained from the numerical integration of the equations up to values of z greater than 0.9. Figs. 3a and 3b show profiles obtained from both the power series approximations and the integration routine. 3a is a plot of normalized electron densities versus normalized spatial coordinate, z, while 3b depicts normalized negative ion densities. The first 100 terms in the power series expansions (eqs. C-6) were used for the series approximations. As is

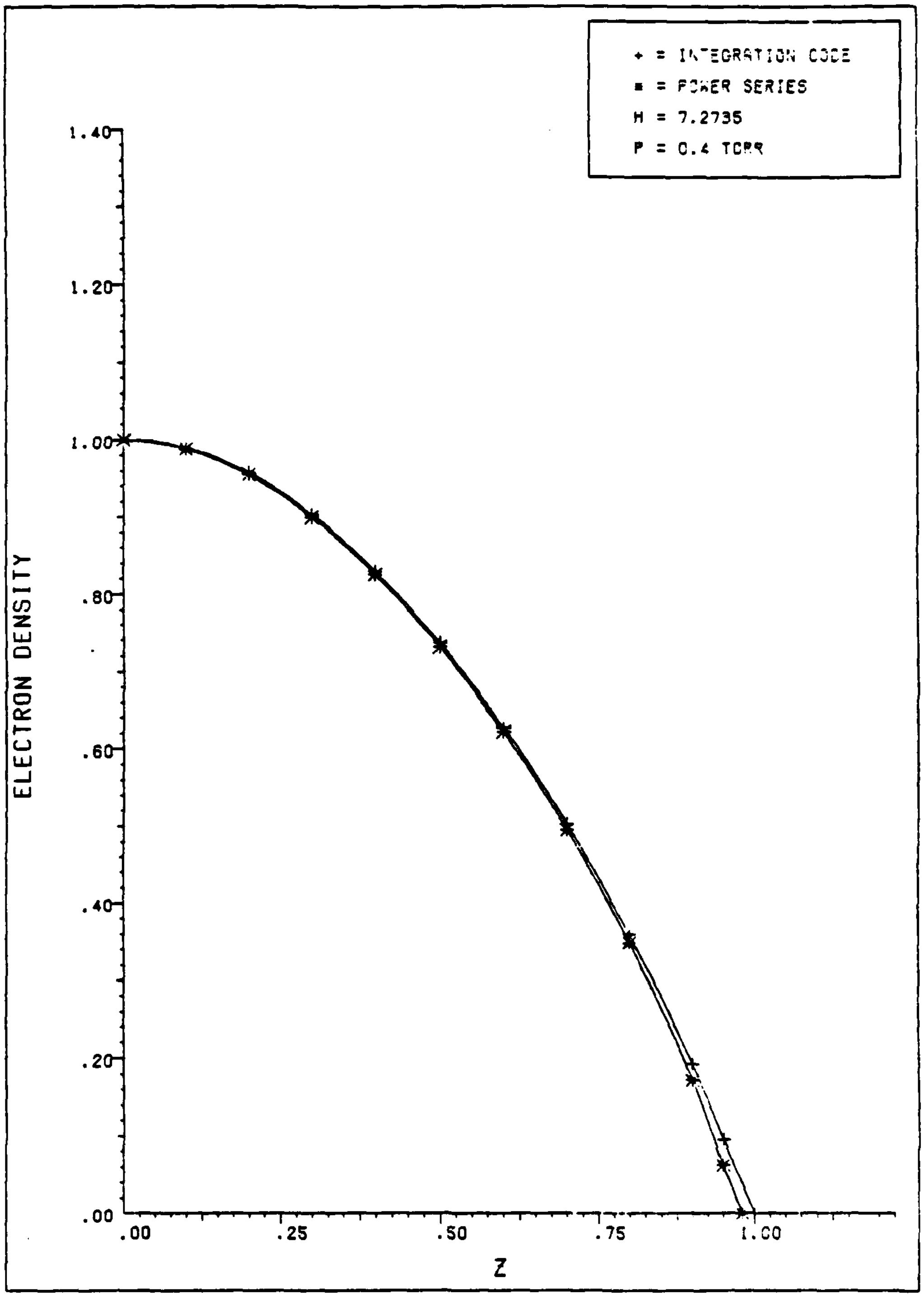

FIG. 3A COMPARISON OF POWER SERIES WITH INT. CODE

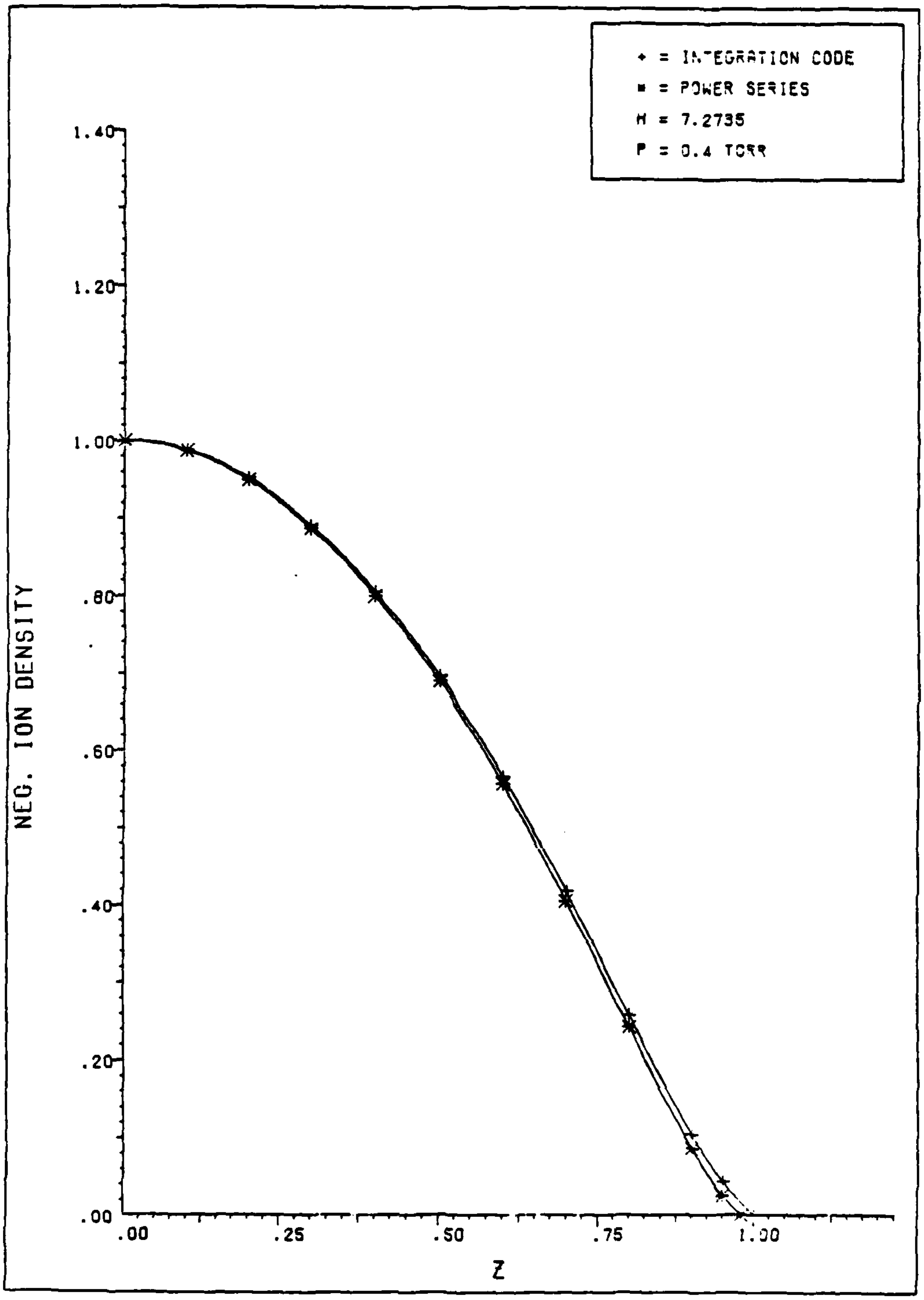

FIG. 3B COMPARISON OF POWER SERIES WITH INT. CODE

evident from the graphs, excellent agreement with the integration routine was obtained. Figs. 4a and 4b demonstrate the convergence behavior of the power series approximations at z=0.9 and z=0.95. The magnitude of the k^{th} term in the power series was plotted as a function of k. In both cases, the magnitudes decreased rapidly for k greater than approximately 90 indicating convergence had been reached. The series does not converge at z=1; thus, the power series approximation begins to diverge between z=.95 and z=1.

A fourth order Runge-Kutta method (5:95) is used for the numerical integration (solution of a second order differential equation). Ingold's parametric solution as expressed in equation B-56 cannot be solved using a Runge-Kutta integration technique since the technique can only handle first-degree differential equations (i.e. the derivatives are not squared, cubed, etc.) This problem can be overcome by recognizing the fact that B-56 can be written as

$$w'' + k^2(Au^s + Bu) = 0 \tag{3-1}$$

where

$$w = \left(A + \frac{2h}{\gamma+1}C\right)u^s + \left(B + \frac{2h}{\gamma+1}D\right)u \tag{3-2}$$

A new problem is encountered in that 3-1 now contains two variables. This problem is dealt with by using an iterative

FIG. 4A POWER SERIES CONVERGENCE (LEE)

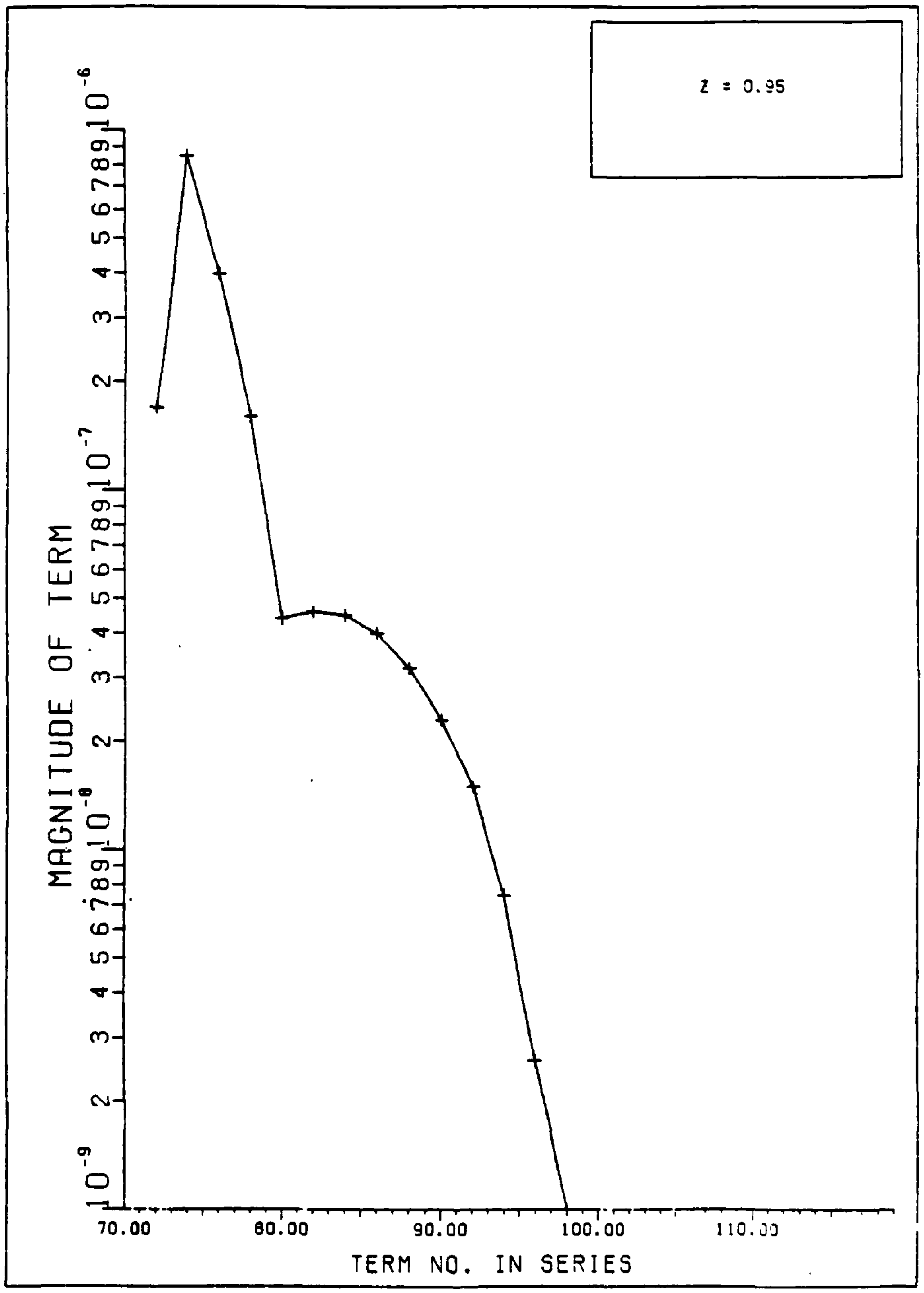

FIG. 4B POWER SERIES CONVERGENCE (LEE)

subroutine that finds the appropriate value of u for a given value of w using equation 3-2. Thus, the subroutine is called whenever a value of x is needed where the last value obtained for w is used to find u.

Mobilities, ionization rates and dissociative attachment rates are taken from Thompson since Ingold doesn't supply any of these values. The numerically generated profiles obtained from the code listed in appendix F did not resemble Ingold's. Rather than obtaining non-proportional profiles as indicated by Ingold, the profiles were seen to be proportional over a wide range of values for h.

Thompson's only analytic expressions are those he obtains for his ambipolar diffusion coefficients. However, it is shown in appendix A that the proportionality relation

$$\frac{\nabla n_-}{\nabla n_e} = \gamma \alpha$$

is necessary to obtain Thompson's diffusion coefficients. From 2-41 we have

$$y = x^{\gamma} \tag{3-3}$$

Equation B-9 is a general relation for which no assumptions specific to Ingold were made, thus, we will use it here and employ 3-3 to obtain

$$x'' + \frac{2\theta_g h}{\theta_e + \theta_g}\left[\gamma(\gamma-1)x^{\gamma-2}x'^2 + \gamma x^{\gamma-1}x''\right] + k^2 x = 0 \tag{3-4}$$

As with Ingold, this can be written as

$$w'' + k^2 x = 0 \tag{3-5}$$

where

$$w = \frac{2\theta_g h}{\theta_e + \theta_g} x^{\gamma} + x \tag{3-6}$$

The coding of these equations for numerical integration proceeded exactly as Ingold's. An iterative subroutine was called when necessary to find the appropriate value of x for a given w using 3-6.

Again a fourth order Runge-Kutta (5:95) method was used for the numerical integration. Thompson uses a Boltzmann distribution to model the negative ion density and concludes that a negative ion temperature of 0.15 ev gives the best agreement between theoretical results and experimental data; thus, θ_g was taken to be 0.15 ev in the numerical code. Thompson also gives γ =16 which fixes θ_e at about 2.4 ev. μ_+ is given as 2.25 cm^2 V^{-1} sec^{-1} (9:517) and Thompson states that μ_-/μ_e =0.0043 and μ_+/μ_e =0.0022 for oxygen. This fixes μ_e at 1022 and μ_- at 4.39 cm^2 V^{-1} sec^{-1}. Using these values for the mobilities and ion

temperatures, it was found that, to obtain profiles similar to those given by Thompson, ionization and dissociative attachment rates on the order of 10 sec^{-1} were needed. The ionization rate is the same order of magnitude as that predicted by the two component Schottky limit where

$$\sqrt{\frac{D_a}{\nu}} = \frac{\pi}{2} \tag{3-7}$$

The above relation yields a value of ν =14.1 sec^{-1} for the mobilities and ion temperatures given by Thompson. If ionization and dissociative attachment rates similar to Lee are used (10^6 and 10^5 sec^{-1}), then mobilities on the same order of magnitude as Lee (10^6 for electrons, 10^3 for ions) are needed to again obtain profiles similar to Thompson's. Exact values for ν and λ are difficult to obtain since Thompson makes no attempt to establish any sort of boundary conditions and none are apparent upon inspection of his profiles. Figs. 5 and 6 show several profiles for various values of ν and λ .

Two Component Limit

A requirement that all solutions to the three component problem should satisfy is that they give results similar to the analytic two component solutions (where only electrons and positive ions are considered) in the two component limit. This limit can be obtained in Lee's code (appendix D), if two variables , h= n_{-0}/n_{e0} and alpha, are reduced to small values.

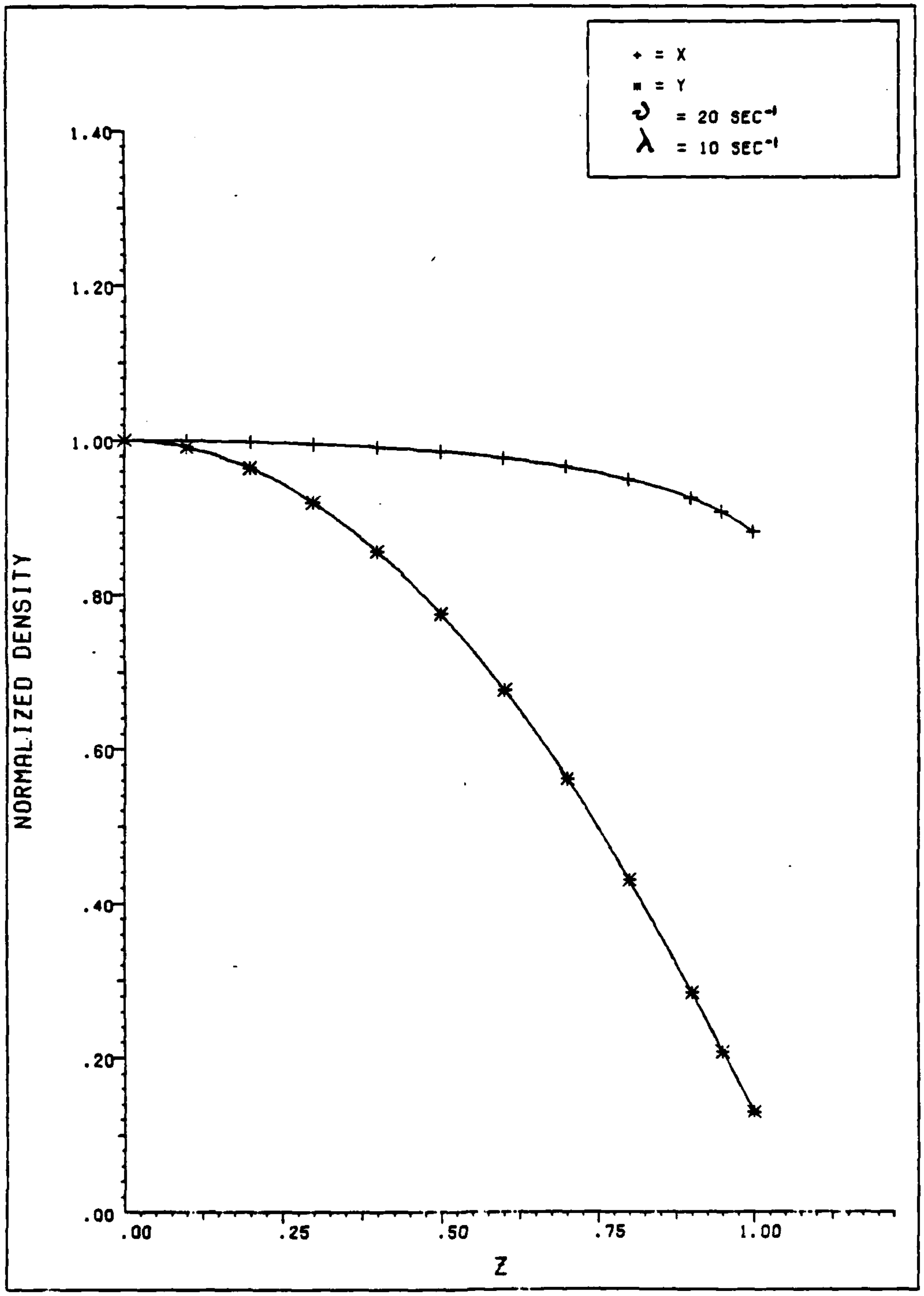

FIG. 5A CHARGED PARTICLE PROFILES (THOMPSON)

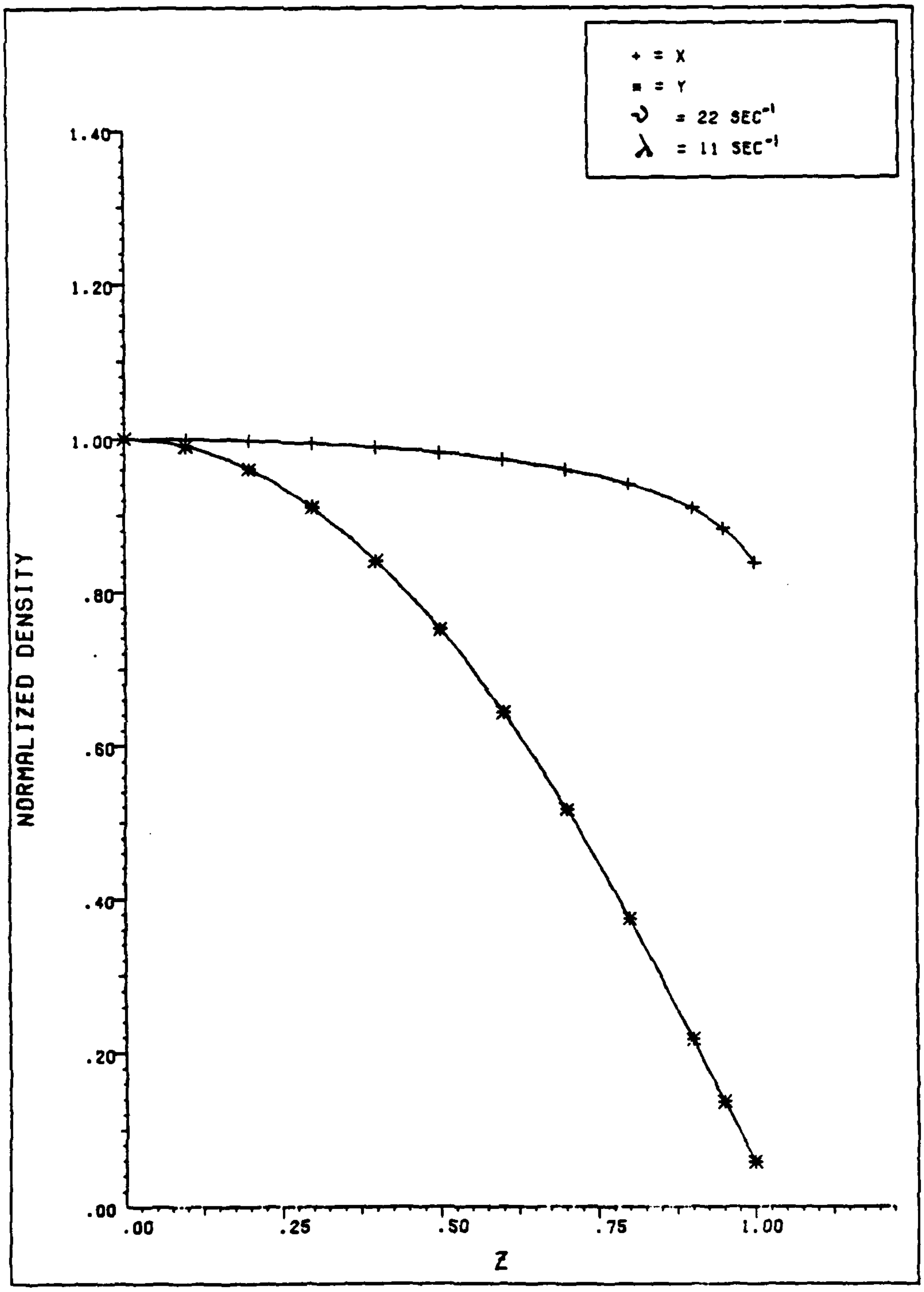

FIG. 5B CHARGED PARTICLE PROFILES (THOMPSON)

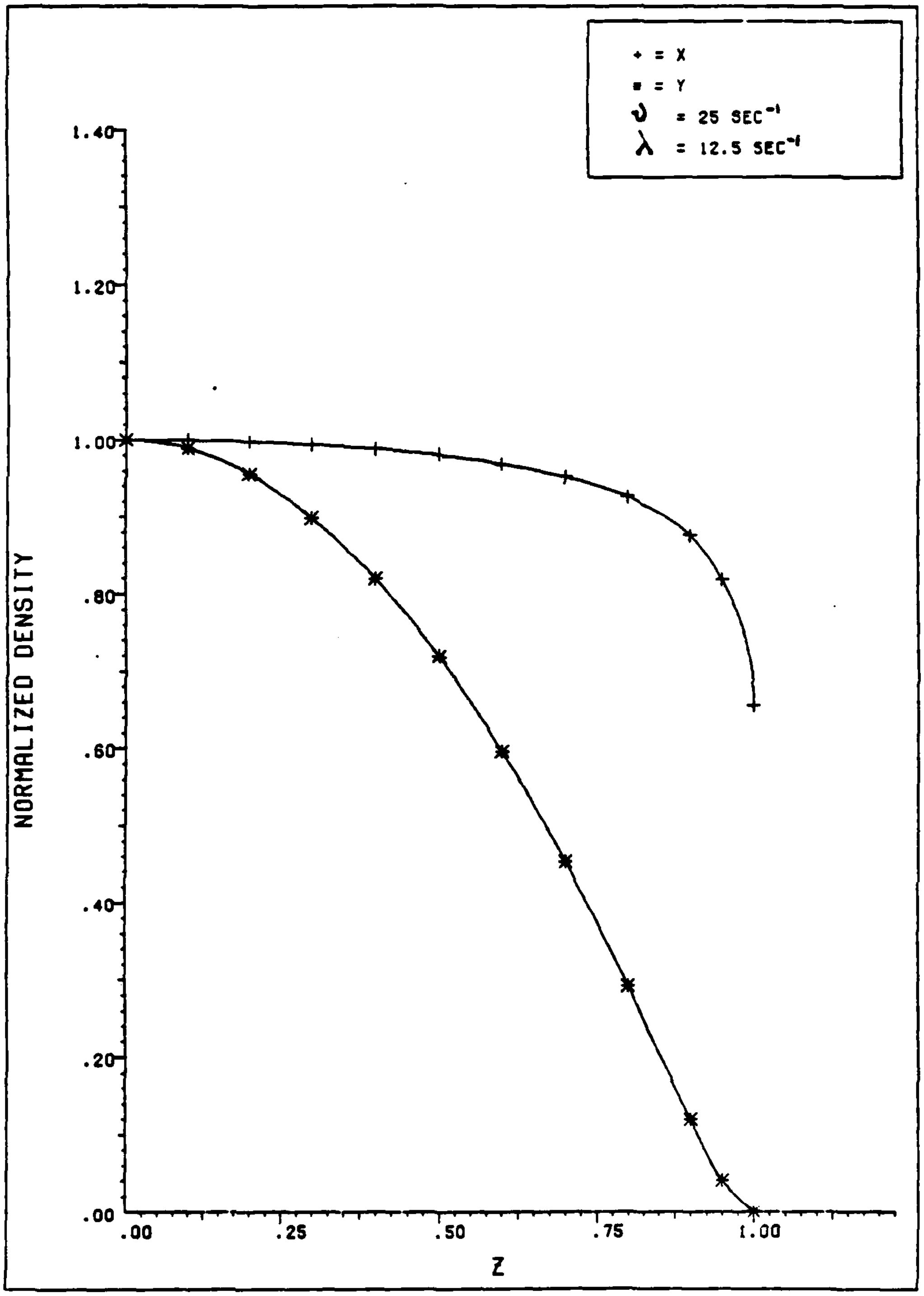

FIG. 6A CHARGED PARTICLE PROFILES (THOMPSON)

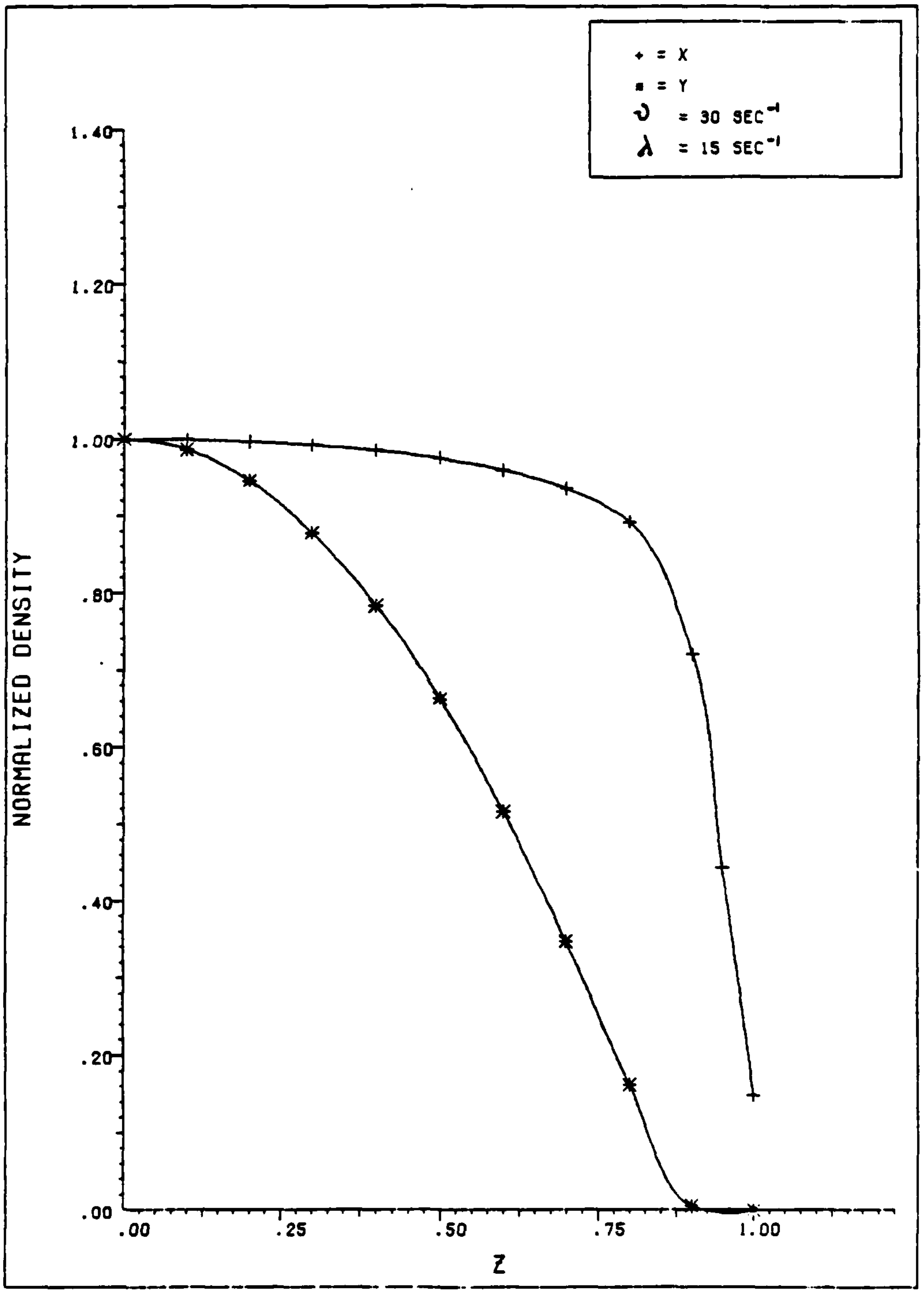

FIG. 6B CHARGED PARTICLE PROFILES (THOMPSON)

We can set h to zero but not alpha since division by zero would occur in a number of places. This problem can be overcome by redefining y_3 and y_4 as

$$y_3 \equiv \frac{j_e(Rz)}{\nu n_{eo} R} \qquad y_4 \equiv \frac{j_-(Rz)}{\nu n_{eo} R} \tag{3-8}$$

rather than the definitions given by C-3. This was done and the appropriate changes were made in the code and it was observed that the profiles did not differ significantly from those obtained by using Lee's definitions and a small value of alpha instead of zero. (Values from 0.1 to 0.001 sec^{-1} gave essentially the same profiles.)

Since Lee uses cylindrical coordinates, the electron profile would be expected to look like a zeroth order Bessel function. We have shown that, in cartesian coordinates, the relation

$$\sqrt{\frac{\nu}{D_a}} = \frac{\pi}{2} \tag{3-9}$$

must be obeyed if the charged particle densities are to go to zero at the wall. Similarly, from our derivation of 3-9, we had obtained

$$\Gamma \equiv -D_a \frac{dn}{dz} \tag{3-10}$$

and

$$\nabla \cdot \Gamma = \nu n \tag{3-11}$$

where, in cylindrical coordinates, we obtain

$$-D_a \frac{1}{z}\frac{d}{dz}\left(z\frac{dn}{dz}\right) = \nu n \qquad (3\text{-}12)$$

or

$$z^2\frac{d^2n}{dz^2} + z\frac{dn}{dz} + \frac{\nu}{D_a}z^2 n = 0 \qquad (3\text{-}13)$$

Thus, if

$$\frac{\nu}{D_a} = 1 \qquad (3\text{-}14)$$

the solution of 3-13 is a zeroth order Bessel function. 3-14 now specifies ν given the mobilities and ion and electron temperatures since

$$\nu = D_a = \frac{\theta_e + \theta_g}{\frac{1}{\mu_e} + \frac{1}{\mu_+}} \qquad (3\text{-}15)$$

Besides including this condition in the code, one other change must be made. The tube radius was assumed to be 1 cm so that the R term in 3-4 could be neglected. We are now forced to include it and set it equal to 2.405 since this is where the first zero of the zeroth order Bessel function occurs. With these changes, it was found that the electron profile given by the code agreed reasonably well with a Bessel function. The code's profile did diverge slightly from the Bessel solution,

increasing to a maximum difference of about 0.05 at about x=0.9 and then converging back to the Bessel solution since both profiles go to zero at the wall.

To see how Ingold's equations reduce in the two component limit, we begin by dividing a, b, c and d by n_{-0} and redefining these terms as a, b, c and d and employing Ingold's relation B-31 to obtain

$$a_0 = \lim_{\alpha, n_- \to 0} a = \frac{\theta_e}{\theta_g}\left(1 + \frac{\theta_g}{\theta_e}\right)\frac{\nu}{\mu_e}\, ; \quad b_0 = \lim_{\alpha, n_- \to 0} b = \frac{\nu}{\mu_+} - \frac{\theta_g}{\theta_e}\frac{\nu}{\mu_e}$$

$$c_0 = \lim_{\alpha, n_- \to 0} c = \frac{\nu}{\mu_+} + \frac{\nu}{\mu_e}\, ; \quad d_0 = \lim_{\alpha, n_- \to 0} d = 0 \tag{3-16}$$

where, again, n_{-0} and α have been set to zero. From B-47 we have

$$f = \frac{a_0}{c_0 - b_0} \tag{3-17}$$

and

$$m = b_0 - d_0 f = b_0\, ; \quad n = c_0 + d_0 f = c_0$$

$$p = d_0 = 0\, ; \quad s = \frac{n}{m} = \frac{c_0}{d_0} \equiv s_0$$

Then, from B-51 and B-53 we have

$$A = \frac{1}{\left(1 - \frac{a_0}{c_0 - b_0}\right)^{c_0/b_0}} \equiv A_0\, ; \quad B = \frac{p}{m-n} = 0$$

$$C = Af = A_0 \frac{a_0}{c_0 - b_0} \equiv C_0\, ; \quad D = 1 + Bf = 1 \tag{3-18}$$

So B-56 becomes

$$A_o s_o u^{s_o-1} u'' + A_o s_o (s_o-1) u^{s_o-2} u'^2 + k^2 A_o u^{s_o} = 0 \tag{3-19}$$

or

$$\frac{d^2}{dz^2}(A_o u^{s_o}) + k^2 A_o u^{s_o} = 0 \tag{3-20}$$

The solution of interest to this equation is

$$A_o u^{s_o} = \cos(kz) \tag{3-21}$$

and from B-50, we see

$$\chi = A_o u^{s_o} = \cos(kz) = \frac{n_e}{n_{eo}} \tag{3-22}$$

From B-55, we have

$$\lim_{\alpha \to 0} k^2 = \frac{\frac{\nu}{\mu_e} + \frac{\nu}{\mu_+}}{\theta_e + \theta_g} = \frac{\nu}{D_a} \tag{3-23}$$

which is the expected result. Thus, Ingold's analytic equations reduce properly in the two component limit.

Likewise, the code in appendix F gives the proper cosine curve for the electron density when n_{-o} and α go to zero and the relation

$$\sqrt{\frac{\nu}{D_a}} = \frac{\pi}{2}$$

is obeyed, even though the code does not give profiles similar to Ingold's in the general, three component case.

For Thompson, we see from 3-4 that as h goes to zero, we obtain

$$\chi'' + k^2 \chi = 0 \tag{3-24}$$

The appropriate solution of which is

$$\chi = \cos(kz) \tag{3-25}$$

where the k expressed here is the same as Ingold's k. Thus, 3-4 also has the proper behavior in the two component limit.

We now turn our attention to Thompson's analytic expressions. As α goes to zero, equation A-3a becomes

$$\begin{aligned} D_+^a &= D_+ \frac{1+\gamma}{1+\mu_+/\mu_e} \\ &= \theta_+ \mu_+ \frac{1+\gamma}{1+\mu_+/\mu_e} = \frac{\theta_+ + \theta_e}{\frac{1}{\mu_+} + \frac{1}{\mu_e}} \end{aligned} \tag{3-26}$$

where we have employed the Einstein relation. We recognize 3-26 as the ambipolar diffusion coefficient for a two component plasma. Also, for A-3c

$$D_e^a = D_+ \frac{1+\gamma}{1+\frac{\mu_+}{\mu_e}} = \frac{\theta_g + \theta_e}{\frac{1}{\mu_+} + \frac{1}{\mu_e}} \tag{3-27}$$

Furthermore, one would expect, in the limit as α goes to infinity, that both A-3a and A-3b would reduce to D_+ if $\mu_+ = \mu_-$. From A-3a, we have

$$\lim_{\alpha \to \infty} D_+^a = D_+ \frac{(2\alpha\gamma)(\alpha \frac{\mu_-}{\mu_e})}{(\alpha\gamma)(\alpha \frac{\mu_+}{\mu_e} + \alpha \frac{\mu_-}{\mu_e})} = D_+ \frac{2 \frac{\mu_+}{\mu_e}}{2 \frac{\mu_+}{\mu_e}} = D_+ \tag{3-28}$$

and from A-3b, we have

$$\lim_{\alpha \to \infty} D_-^a = D_+ \frac{1}{\gamma} \frac{\mu_-}{\mu_e} \frac{2\alpha\gamma}{\alpha \frac{\mu_+}{\mu_e} + \alpha \frac{\mu_-}{\mu_e}} = D_+ \tag{3-29}$$

Thompson also gives an analytic expression for the electric field as follows (10:20):

$$\frac{E(\alpha)}{E(0)} = \frac{1 - D_+/D_+^a}{1+\gamma} \frac{1+\gamma}{\gamma} \tag{3-30}$$

where E(0) is the electric field for $\alpha = 0$. Thus, if we look at 3-30 in the limit as α goes to zero, we would expect the right side to go to one. Doing this, we obtain

$$\lim_{\alpha \to 0} \frac{E(\alpha)}{E(0)} = \frac{1 - \frac{1+\frac{\mu_+}{\mu_e}}{1+\gamma}}{1} \left(\frac{1+\gamma}{\gamma}\right) = \frac{1+\gamma-1-\frac{\mu_+}{\mu_e}}{\gamma} = \frac{\gamma - \frac{\mu_+}{\mu_e}}{\gamma} \tag{3-31}$$

Thus, Thompson's expression for the electric field does not

reduce properly in the two component limit. Failing to meet this basic requirement, we shall conclude that Thompson's electric field expression is incorrect. An attempt was made to derive a correct analytic expression, but none could be found that did not include a term similar to $\nabla n_e / n_e$.

Sensitivity to Parameters

Lee's profiles were seen to be very sensitive to h, the on axis ratio of negative ions to electrons. Many difficulties were encountered in trying to find a value of h which gives the best fit to the boundary conditions. A measure of the fit to the boundary conditions at the wall is given by the terms D1F1 and D1F2 as found in the code outlined in appendix D. D1F1 represents the difference between y_3 , as found by the integration routine and y_3 , as given by C-25. D1F2 represents the difference between y_4 , as found by the integration routine and y_4 , as given by C-25. The fit is made at a point a small distance, ϵ_2 , away from the wall. Thus, h must be chosen so as to minimize D1F1 and D1F2.

Because of the poor convergence, Lee's graphs of pressure versus electron temperature and h versus pressure (L:4702) were used to obtain initial values. It was found that if h was off by more than several tenths, y_2 would begin to rise before the wall could be reached. Once a range for h was found where y_2' was always negative, the values where D1F1 and D1F2 reached minimum values, or changed sign, could be investigated with a smaller Δh . To complicate the problem, though, convergence

was not steady and several apparent minima often occurred, even with spacings as small as 0.01. Upon closer investigation of some minima, using a smaller Δh , unusual behavior such as γ_2' going positive would occur. Well behaved convergence could only be achieved when the initial guess wasn't more than about 0.1 away from the final value.

For P=0.4 torr, it was found that θ_e =2.1 eV and h=7.2735. This gave minimum values of D1F1 and D1F2 of -0.879 and 0.587 respectively. Increasing h by 0.01 causes D1F1 and D1F2 to rise to -2.253 and 1.288 and increasing h by 0.1 causes D1F1 and D1F2 to jump to -40.508 and 23.206.

Increasing θ_e by 0.01 eV resulted in an increase in h to 7.4743 with D1F1 and D1F2 remaining at about the same values as when θ_e =2.1 eV. Increasing θ_e by 0.1 eV to 2.2 led to minimums of D1F1 and D1F2 on the order of 10^2 .

Finally, pressure was varied and it was found that smaller values of D1F1 and D1F2 could be obtained for P=0.47 torr when θ_e =2.1 eV. It was found that D1F1=0.0135 and D1F2=-0.0083 for h=7.4199. It is possible that D1F1 and D1F2 could be reduced to even smaller values given the right combination of pressure, electron temperature and h, but, given the difficulty of trying to simultaneously minimize three parameters and the fact that h does not converge steadily, time limitations prevented further investigation.

Figures 5 and 6 demonstrate the sensitivity of Thompson's profiles to changes in ν and λ . One might expect that

changes in θ_e would lead to steeper negative ion profiles since $y = x^\gamma$, but, it was observed that an increase in θ_e resulted in flatter electron profiles and relatively no change in the negative ion profiles. Even with θ_c increased to 10 times the value used by Thompson and the code in appendix E, it was seen that the negative ion profile changed by less than 0.01 where, for ν =22 and λ =11, the electron profile was almost flat, reaching a minimum of about 0.98 at the wall. Decreasing θ_g resulted in both steeper electron and negative ion profiles. (For θ_g =0.05 eV, x went to zero at 0.88 and y went to zero at 0.61.)

The sensitivity to changes in parameters in the code for Ingold was not investigated since it would have been pointless considering all the problems previously indicated in its developement.

IV. RESULTS AND CONCLUSIONS

Comparison of Results

As was mentioned earlier, if Thompson's data for mobilities, electron and ion temperatures and ratio of negative ion density to electron density on axis are used in the code for Thompson, it is found that ionization and dissociative attachment rates on the order of 10 are needed to give profiles that are similar to those given by Thompson's experiemental data (10:820). Lee, on the other hand, gives empirical expressions for ionization and dissociative attachment rates and mobilities that depend on electron temperature, pressure and pressure respectively. Using electron temperatures similar to Thompson, and pressures of about 0.4 torr, we obtain ionization and dissociative attachment rates on the order of 10^6 and 10^5. Thompson indicates that his measurements were taken at 0.04 torr, so one would expect smaller ionization and dissociative attachment rates, however, since both are directly proportional to the pressure, one would not expect numbers as small as are indicated by the code.

Likewise, there is a considerable difference in the mobilities used in each paper. Lee's are about three orders of magnitude larger than Thompson's. Again, this seems inconsistant in that one would expect smaller mobilities with Lee since he uses larger pressures than Thompson.

If, however, the appropriate changes are made in Lee and Thompson, the profiles produced by the code for Lee begin to

take the appearance of the profiles produced by the code for Thompson. The changes necessary to do this are: 1. Lee's equations must be changed from cylindrical coordinates to Cartesian coordinates. 2. Lee's mobilities and ionization and dissociative attachment rates must be used in Thompson. 3. Associative detachment must be ignored in Lee. 4. The same value of h must be used in both codes. These changes put both codes in cartesian coordinates and start both off with the same set of parameters. The difference between the profiles found from the Lee and Thompson codes decreases as h increases. At h=40, the difference between both the electron and negative ion profiles found from the two codes was less than about 3% throughout the normalized tube radius. Figs. 7 and 8 show the results for h=10 and 40. Profiles do not extend to the wall (z=1) because no variation of parameters was performed since the only purpose of the comparison was to see whether the two codes gave similar profiles given the same initial parameters. Without associative detachment, then, Lee and Thompson seem to agree reasonably well, however, keeping associative detachment in Lee and including it in Thompson does not give similar profiles.

Boundary Conditions

Boundary conditions play an important role in determining the exact behavior of the charged particle profiles. Requiring the fluxes, electric fields and gradients of the charged particles to be zero on axis (z=0) seems to be reasonable when

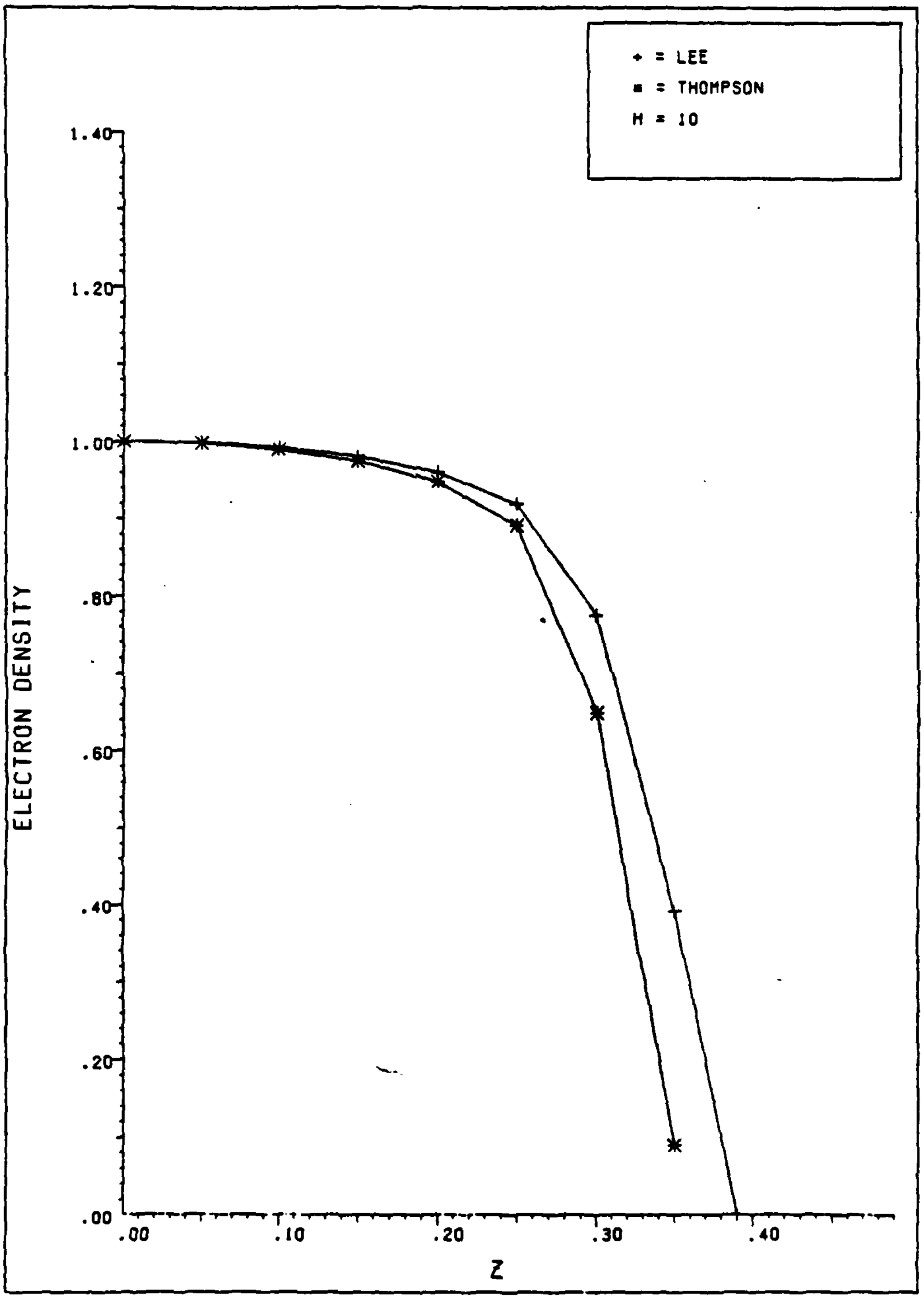

FIG. 7A COMPARISON OF LEE AND THOMPSON

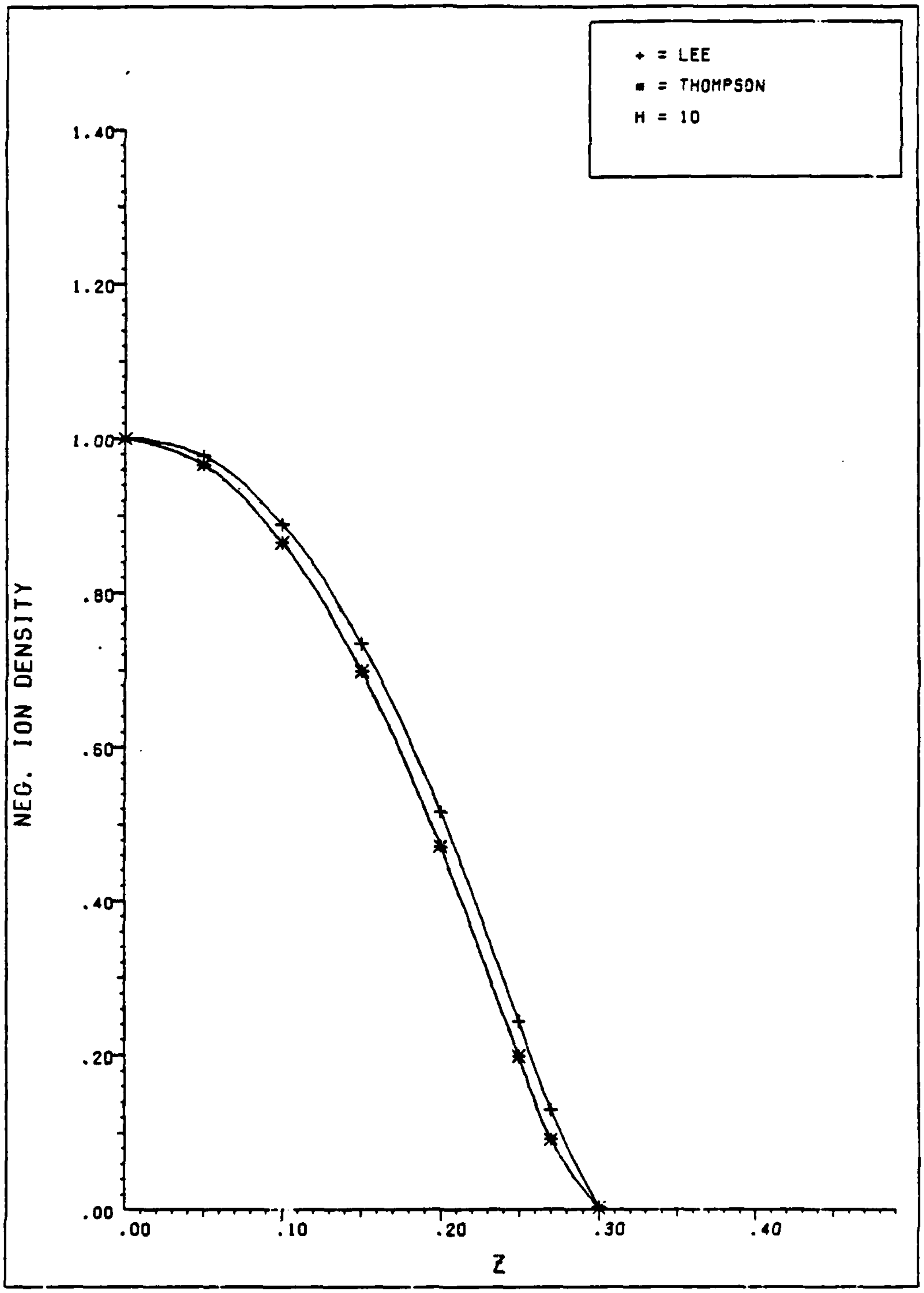

FIG. 7B COMPARISON OF LEE AND THOMPSON

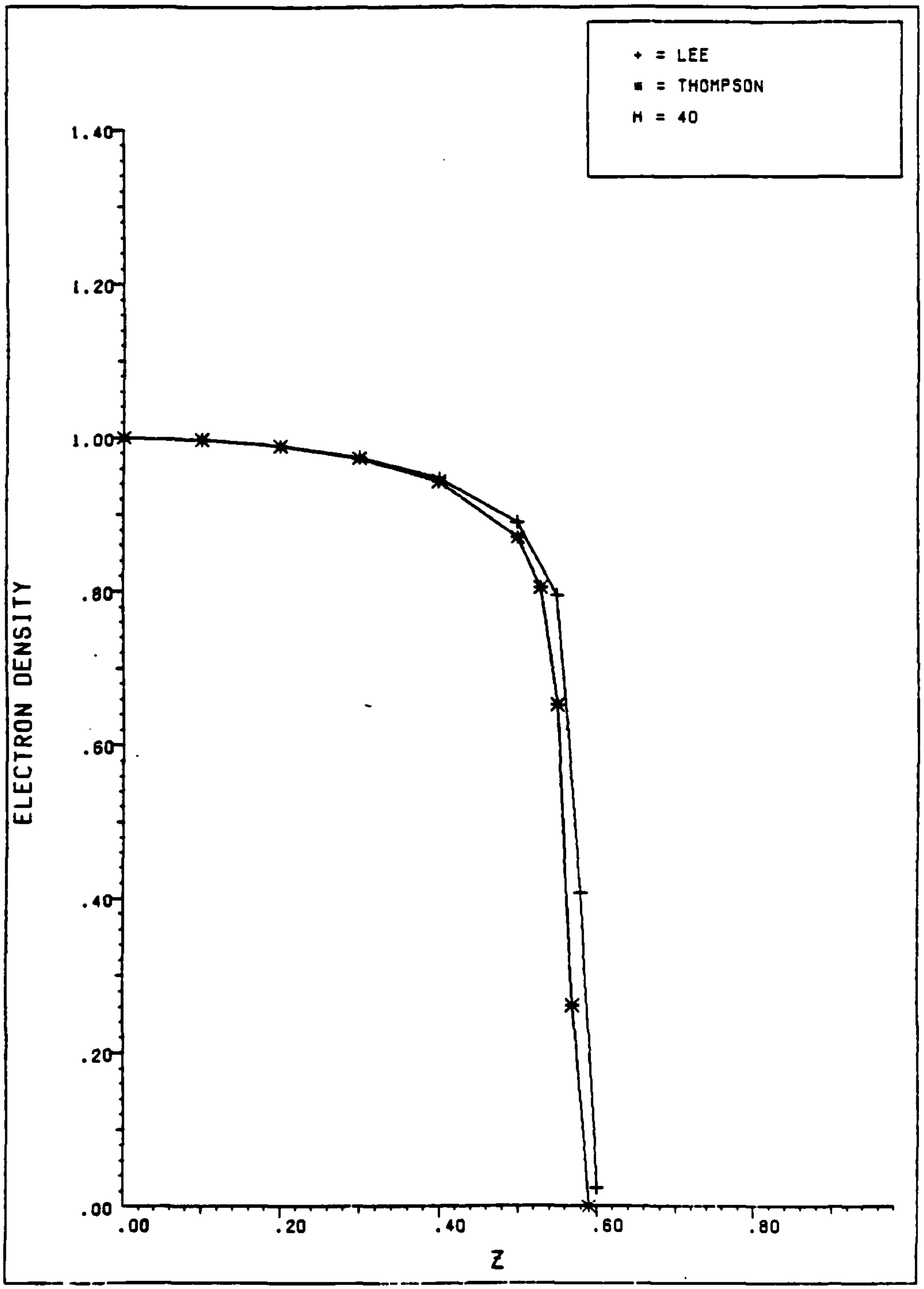

FIG. 8A COMPARISON OF LEE AND THOMPSON

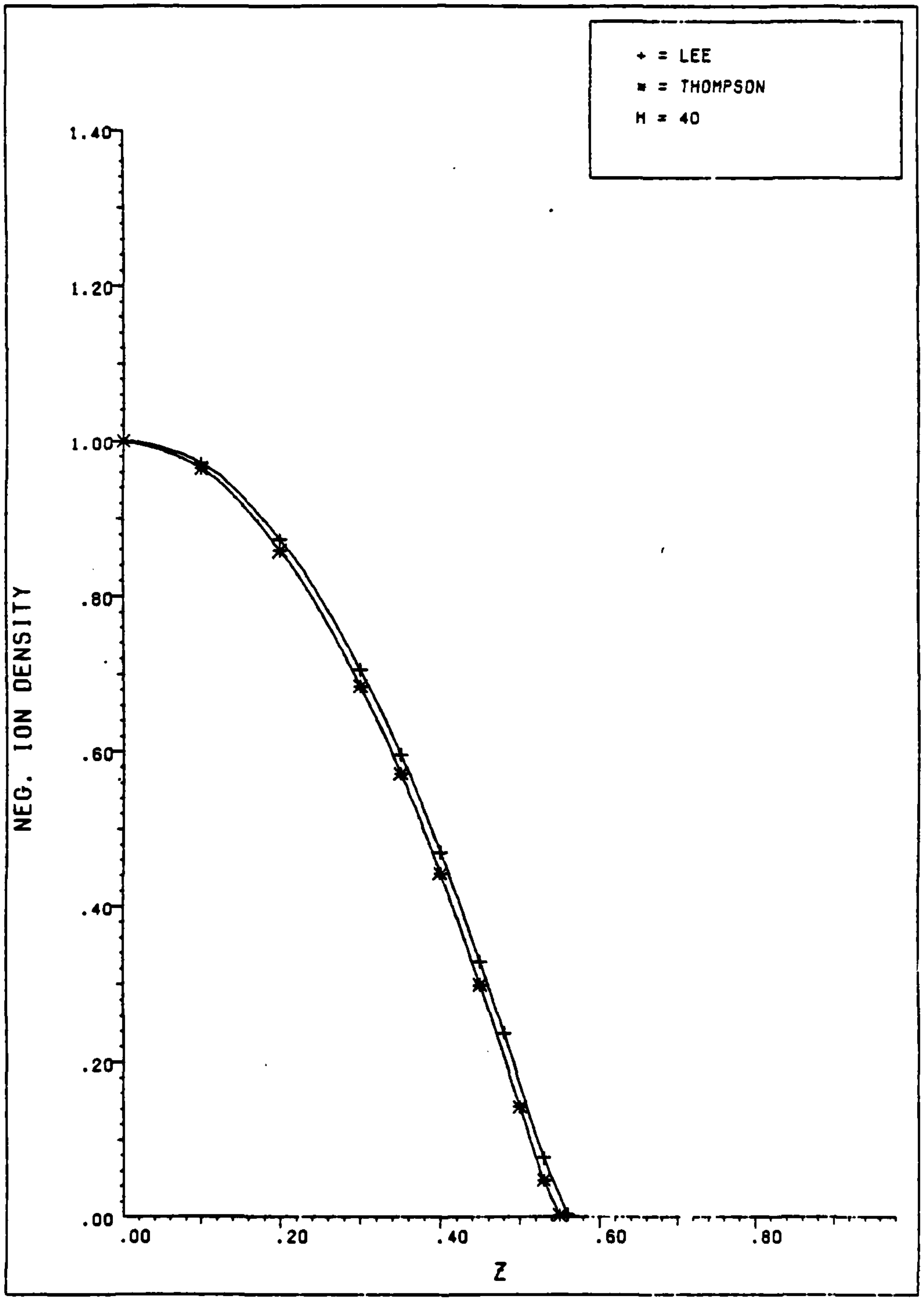

FIG. 8B COMPARISON OF LEE AND THOMPSON

one considers the symmetry of the problem. Ingold and Lee, however, further constrain the problem by requiring the particle densities to go to zero at the wall with the assumption of quasineutrality valid over the entire region. Another such boundary condition is certainly necessary to adequately solve the problem, however, there has been debate as to whether the condition of zero charge density and quasineutrality at the wall is appropriate.

Forest and Franklin (Ref. 3) and Edgley and von Engle (Ref. 2) argue that a sheath region must exist at the wall where quasineutrality cannot be obeyed. The reason for this is that, in the sheath region, electron diffusion is governed by the electrons thermal velocity which is much greater than both the ion thermal velocity and the ion drift velocity. Thus, once inside the sheath region, electrons will diffuse to the wall much faster than positive ions. So, to obtain equal fluxes at the wall, there must be more positive ions in the sheath region than electrons. This arguement seems to preclude the existance of negative ions, which is understandable for Forest and Franklin since they only consider the two component case. The reason is unclear, though, for Edgley and von Engle. Their results show that the negative ions migrate toward the axis and very few make it to the wall, but use of this knowledge in setting up the initial boundary condition would be an inappropriate approach to the problem.

For the sake of arguement, we shall assume the approximation

is valid and continue. Thus, at the wall, Edgley and von Engle can equate the electron drift velocity to the electron's thermal velocity since the electrons are close enough to reach the wall through the random motion of thermal velocity. Using simple kinetic theory, Edgley and von Engle write (2:379)

$$U_e = \frac{v_e}{c_s} = \frac{\Gamma_+ - \Gamma_-}{\gamma} = \left(\frac{2}{\pi}\right)^{1/2} \left(\frac{kT_e}{m_e}\right)^{1/2} \Big/ \left(\frac{kT_e}{m_+}\right)^{1/2} = \left(\frac{2}{\pi}\,\frac{m_+}{m_e}\right)^{1/2} \qquad (4\text{-}1)$$

where c_s =positive ion sound speed, v_e = electron drift velocity, y= n_-/n_{-0} , Te= electron temperature and m= mass.

Edgley and von Engle perform an integration routine similar to Lee except they do not assume quasineutrality and, thus, have seven equations in seven unknowns. Equation 4-1 provides a boundary condition on the normalized electron velocity at the wall. Integration of the equations proceeds to the wall where the normalized electron velocity must be equal to the ratio of the particle masses as given in 4-1. If this condition is not satisfied, parameters are varied, as was done for Lee, until the boundary condition is met.

One possible criticism to this approach is the use of 4-1 as a boundary condition on the equations governing the physics throughout the rest of the plasma. Outside of the plasma sheath, the drift velocity of electrons is governed by outward diffusion and an electric field created by a small deviation from strict charge neutrality. Near the wall, though, the drift velocity out of the plasma is governed by the electron's thermal

velocity. Thus, the equations which govern the physics of the sheath are different than those which apply throughout the rest of the plasma, yet von Engle and Edgley use the sheath condition at the wall as a boundary condition on the equations used in the integration routine.

A question also arises as to whether the plasma sheath is worthy of consideration at all. Typically, the thickness of a plasma sheath is on the order of a Debye length, where, beyond this distance the plasma remains undisturbed. From Ref. 8 we have the following expression (8:71)

$$\lambda_d = \sqrt{\epsilon_0 k T / n_e e^2} \qquad (4\text{-}2)$$

where λ_d is the Debye length and n_e is the electron number density. Using Lee's numbers, we find that the number of oxygen molecules per cubic centimeter is on the order of 10^{15} . Assuming the ratio of electrons to oxygen molecules to be no less than 1 to 100000, we obtain a maximum value for λ_d of about 0.01 cm. This represents only 1/100 the radius of a 1 cm tube.

Conclusion

Three main conclusions can be drawn from this presentation. First, if the same parameters and coordinate systems are used in both Lee and Thompson, and associative detachment is ignored, then Lee's profiles looked like Thompson's; but, when associative detachment is included in both, the two are not

similar. This may indicate that Thompson's relation is valid only in limiting circumstances (e.g. when associative detachment can be ignored). Thompson's experimental measurements are taken at about 1/10 of the pressures Lee uses for his profiles and, since Lee states that the associative detachment rate is proportional to the oxygen density and negative ion density, it is possible that associative detachment was not a major factor in Thompson's experiments. Some indication as to where Thompson and Lee begin to predict significantly different results can be found by comparing the two codes when associative detachment is included in Lee but not in Thompson. This was done for different values of pressure and h and the results are given in the table on the following page. The difference between the normalized electron densities and the normalized negative ion densities predicted by each code were evaluated at each of the 100 mesh points (except the first and last). The maximum difference in the normalized electron densities is given by MAXX. MAXY represents the maximum difference in the negative ion densities. DX and DY represent the average difference in the densities. X and Y are the normalized electron and negative ion density, respectively, at the wall, as given by Thompson's code. Values of h larger than about 10 resulted in rising negative ion profiles as calculculated in the Lee code.

Secondly, the fact that the power series approximation used by Lee near z=0 gave excellent agreement with the integration routine out to greater than z=0.9 indicates that numerical

COMPARATIVE DIVERGENCE OF THOMPSON WITH LEE

H	P(torr)	DX(10^{-4})	DY(10^{-4})	MAXX(10^{-2})	MAXY(10^{-2})	X	Y
2	.02	.831	18.2	.185	3.86	.99	.86
2	.04	2.62	66.7	.482	13.54	.96	.48
2	.06	3.16	116.	1.12	18.37	.85	.07
2	.08	19.6	117.	5.12	18.8	.59	.00
5	.02	1.08	18.1	.238	3.92	.99	.94
5	.04	4.00	71.3	.843	15.4	.98	.75
5	.06	7.35	158.	1.28	33.0	.95	.44
8	.02	1.19	17.6	.261	3.84	1	.96
8	.04	4.52	70.4	.970	15.33	.99	.83
8	.06	9.20	157.	1.85	34.2	.97	.63

integration of equations C-1 probably isn't necessary. The Taylor series expansion used near the wall could also be used in conjunction with the power series since the power series begins to diverge near the wall. A possible problem may occur in that Lee only keeps the first term of the Taylor series expansion, thus severely limiting the distance from the wall where the approximation is valid. This can be overcome by writing the expansions, C-24, as infinite series and developing recursion relations similar to C-11, C-14, C-16 and C-23. This was done, but a converging series could not be obtained. This may be due to a simple algebraic error, but time limitations prevented further work on this area.

Finally, we note the numerous problems that exist with Ingold's work. The fact that B-5 can not be properly established casts doubt on his entire development since it is a cornerstone of his work. Also, it was shown that a completely different solution could be obtained without constraining the problem any more than does Ingold. This seems to indicate that Ingold's solution may be somewhat arbitrary. Also, as was pointed out earlier, the numerical solution developed for Ingold did not give profiles similar to Ingold's. This may indicate an error was made in the development of the numerical solution, but the fact that the code gave the proper results in the two component limit (as did Ingold's analytic equations) serves as an argument against that. In fact, all numerical solutions and analytic equations reduced properly in the two component limit

except Thompson's relation for the electric field (3-30).

Recommendations

Many areas of this study could benefit from further work. Among these is the re-developed solution to Ingold, done in chapter 2, using the relation

$$X''(ax'+by')=y''(cx'+dy')$$

If another boundary condition at the wall could be found, then numerical profiles could be developed in a manner similar to the method employed in appendix F. Also the alternate solution, 2-32, cold be solved analytically to give charged particle profiles.

Further work could also be done to finish the developement of an infinite Taylor series expansion at the wall as described earlier. This could then be coupled with Lee's power series expansions to give a complete solution throughout the entire discharge tube, eliminating the need for numerical integration. It may not even be necessary to use terms higher than first order in the Taylor series expansions since the power series expansions were seen to converge out to z=.95, but this assumption should not be made without further investigation.

An important final recommendation is to compare the various approaches with actual experimental data since a theory isn't worth much if it falls short of reality.

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APPENDIX A

Derivation of Thompson's equations

Thompson begins with the diffusion equations

$$\Gamma_+ = -D_+\nabla n_+ + \mu_+ n_+ E \tag{A-1a}$$

$$\Gamma_- = -D_-\nabla n_- - \mu_- n_- E \tag{A-1b}$$

$$\Gamma_e = -D_e\nabla n_e - \mu_e n_e E \tag{A-1c}$$

which he writes as

$$\Gamma_+ = -D_+^a \nabla n_+ \tag{A-2a}$$

$$\Gamma_- = -D_-^a \nabla n_- \tag{A-2b}$$

$$\Gamma_e = -D_e^a \nabla n_e \tag{A-2c}$$

by defining ambipolar diffusion coefficients as follows:

$$D_+^a = D_+\left[\frac{(1+\gamma+2\alpha\gamma)(1+\alpha\mu_-/\mu_e)}{(1+\alpha\gamma)(1+\mu_+(1+\alpha)/\mu_e+\alpha\mu_-/\mu_e)}\right] \tag{A-3a}$$

$$D_-^a = D_+\left[\frac{1}{\gamma}\frac{\mu_-}{\mu_e}\frac{1+\gamma+2\alpha\gamma}{1+\mu_+(1+\alpha)/\mu_e+\alpha\mu_-/\mu_e}\right] \tag{A-3b}$$

$$D_e^a = D_+\left[\frac{1+\gamma+2\alpha\gamma}{1+\mu_+(1+\alpha)/\mu_e+\alpha\mu_-/\mu_e}\right] \tag{A-3c}$$

To obtain A-3, we begin by stating the assumptions made by

Thompson, namely, $\Gamma_+ = \Gamma_- + \Gamma_e$ and $n_+ = n_- + n_e$. Therefore, A-1a can be written as

$$\Gamma_- + \Gamma_e = -D_+ \nabla n_+ + \mu_+ n_+ E \qquad \text{(A-4)}$$

Subtracting A-1b from A-4 we have

$$\Gamma_e = D_- \nabla n_- + \mu_- n_- E - D_+ \nabla n_+ + \mu_+ n_+ E \qquad \text{(A-5)}$$

Multplying A-5 by $\mu_e n_e$ and A-1c by $\mu_+ n_+$ and adding the two, we have

$$(\mu_- n_- + \mu_+ n_+)\Gamma_e = -D_e(\mu_- n_- + \mu_+ n_+)\nabla n_e - \mu_e n_e(\mu_- n_- + \mu_+ n_+)E \qquad \text{(A-6)}$$

Dividing through by $\mu_e n_e$ the equation becomes

$$\frac{\mu_-}{\mu_e}\alpha + 1 + \frac{\mu_+}{\mu_e}(\alpha+1))\Gamma_e = D_- \nabla n_- - D_+ \nabla n_+ - D_e\left(\frac{\mu_-}{\mu_e}\alpha + \frac{\mu_+}{\mu_e}(\alpha+1)\right)\nabla n_e \qquad \text{(A-7)}$$

Rearranging and factoring out a $-D_+ \nabla n_e$ we have

$$\Gamma_e = -D_+ \nabla n_e \left[\frac{-\frac{D_-}{D_+}\frac{\nabla n_-}{\nabla n_e} + \frac{\nabla n_+}{\nabla n_e} + \frac{D_e}{D_+}\left(\frac{\mu_-}{\mu_e}\alpha + \frac{\mu_+}{\mu_e}(\alpha+1)\right)}{1 + \mu_+(1+\alpha)/\mu_e + \alpha\mu_-/\mu_e} \right] \qquad \text{(A-8)}$$

Factoring out μ_-/μ_+ we have

$$\Gamma_e = -D_+ \nabla n_e \left[\frac{-\frac{D_-}{D_+}\frac{\mu_+}{\mu_-}\frac{\nabla n_-}{\nabla n_e} + \frac{\mu_+}{\mu_-}\frac{\nabla n_+}{\nabla n_e} + \frac{D_e}{D_+}\left(\frac{\mu_+}{\mu_e}\alpha + \frac{\mu_+^2}{\mu_e\mu_-}(\alpha+1)\right)}{1 + \mu_+(1+\alpha)/\mu_e + \alpha\mu_-/\mu_e}\frac{\mu_-}{\mu_+} \right] \qquad \text{(A-9)}$$

Employing the Einstein relation, 1-8, and assuming $\nabla n_+ = \nabla n_- + \nabla n_e$ the equation becomes

$$\Gamma_e = -D_+ \nabla n_e \left[\frac{\left(-\frac{\nabla n_-}{\nabla n_e} + \left(\frac{\nabla n_-}{\nabla n_e} + 1 \right) \frac{\mu_+}{\mu_-} + \gamma\alpha + \frac{\mu_+}{\mu_-} \gamma(\alpha+1) \right) \frac{\mu_-}{\mu_+}}{1 + \mu_+(1+\alpha)/\mu_e + \alpha \mu_-/\mu_e} \right] \tag{A-10}$$

To obtain eq. A-3c, we must require the numerator in A-10 to sum to $1 + \gamma + 2\alpha\gamma$. This can be done one of two ways: the first is to assume $\mu_+ = \mu_-$ which is not always true. If the assumption is not made, we have

$$-\frac{\mu_-}{\mu_+} \frac{\nabla n_-}{\nabla n_e} + \frac{\nabla n_-}{\nabla n_e} + 1 + \frac{\mu_-}{\mu_+} \gamma\alpha + \gamma(\alpha+1) = 1 + \gamma + 2\alpha\gamma$$

or

$$\frac{\nabla n_-}{\nabla n_e} = \gamma\alpha \tag{A-11}$$

With this proportionality relation, A-10 reduces to A-2c.

To obtain A-2b, we begin by subtracting A-1c from A-4 to get

$$\Gamma_- = -D_+ \nabla n_+ + \mu_+ n_+ E + D_e \nabla n_e + \mu_e n_e E \tag{A-12}$$

Multiplying A-12 by $\mu_- n_-$ and A-1b by $\mu_+ n_+ + \mu_e n_e$ and adding we have, after dividing by $\mu_e n_e$,

$$\Gamma_- \left(1 + \mu_+(1+\alpha) + \frac{\mu_-}{\mu_e}\alpha\right) = -\frac{\mu_-}{\mu_e} \alpha D_+ \nabla n_+ - \left(\frac{\mu_+}{\mu_e}(\alpha+1) + 1 \right) D_- \nabla n_- + \frac{\mu_+}{\mu_e} \alpha D_e \nabla n_e \tag{A-13}$$

Rearranging and factoring out $-D_+\nabla n_-\left(\frac{1}{\gamma}\frac{\mu_-}{\mu_e}\right)$ we obtain

$$\Gamma_- = -D_+\nabla n_-\left[\frac{1}{\gamma}\frac{\mu_-}{\mu_e}\,\frac{\alpha\gamma + \frac{\nabla n_e}{\nabla n_-} + \gamma + \frac{\mu_e}{\mu_+}\gamma + 2\alpha\gamma - \frac{\mu_e}{\mu_+}\gamma^2\alpha\frac{\nabla n_e}{\nabla n_-}}{1 + \mu_+(1+\alpha) + \mu_-\alpha/\mu_e}\right]$$

where the Einstein relation and the fact that $\nabla n_+ = \nabla n_e + \nabla n_-$ has again been used. Again we want the numerator to reduce to $1+\gamma+2\alpha\gamma$. Thus,

$$\alpha\gamma\frac{\nabla n_e}{\nabla n_-} + \gamma + \frac{\mu_e}{\mu_+}\gamma + 2\alpha\gamma - \frac{\mu_e}{\mu_+}\gamma^2\alpha\frac{\nabla n_e}{\nabla n_-}$$

$$= \frac{\nabla n_e}{\nabla n_-}\left(\alpha\gamma + \frac{\mu_e}{\mu_+}\gamma^2\alpha\right) + \gamma - \frac{\mu_e}{\mu_+}\gamma + 2\alpha\gamma$$

$$= 1 + \gamma + 2\alpha\gamma$$

or

$$\frac{\nabla n_e}{\nabla n_-}\gamma\alpha\left(1+\gamma\frac{\mu_e}{\mu_+}\right) = 1 + \frac{\mu_e}{\mu_+}\gamma \qquad \text{(A-14)}$$

At this point, we can see that assuming $\mu_+ = \mu_-$ will not provide a solution. Instead, we obtain the same relation previously found, i.e. $\frac{\nabla n_-}{\nabla n_e} = \gamma\alpha$.

To establish A-2a, we add A1-b to A-1c and multiply through by $\mu_+ n_+$ to get

$$\mu_+ n_+(\Gamma_e + \Gamma_-) = \mu_+ n_+ \Gamma_+ = -D_e\mu_+ n_+\nabla n_e$$
$$-\mu_e n_e \mu_+ n_+ E - D_-\mu_+ n_+ \nabla n_- - \mu_-\mu_+ n_+ \nabla n_- E \qquad \text{(A-15)}$$

Multiplying A-1a by $(\mu_- n_- + \mu_e n_e)$ and adding to A-15 we obtain

$$(\mu_+ n_+ + \mu_e n_e + \mu_- n_-)\Gamma_+ = -D_e \mu_+ n_+ \nabla n_e - D_- \mu_+ n_+ \nabla n_- - D_+(\mu_e n_e + \mu_- n_-)\nabla n_+ \tag{A-16}$$

Rearranging, dividing through by $\mu_e n_e$, and factoring out $-D_+ \nabla n_+$, we have

$$\Gamma_+ = -D_+ \nabla n_+ \left[\frac{\frac{D_e}{D_+}\frac{\mu_+}{\mu_e}\frac{n_+}{n_e}\frac{\nabla n_e}{\nabla n_+} + \frac{D_-}{D_+}\frac{\mu_+}{\mu_e}\frac{n_+}{n_e}\frac{\nabla n_-}{\nabla n_+} + 1 + \frac{\mu_- n_-}{\mu_e n_e}}{1 + \mu_+(1+\alpha) + \frac{\mu_-}{\mu_e}\alpha} \right] \tag{A-17}$$

Again, using the Einstein relation and the fact that $n_+ = n_- + n_e$ and $\nabla n_+ = \nabla n_- + \nabla n_e$, we obtain

$$\Gamma_+ = -D_+ \nabla n_+ \left[\frac{\gamma(1+\alpha)\frac{\nabla n_e}{\nabla n_- + \nabla n_e} + \frac{\mu_-}{\mu_e}(1+\alpha)\frac{\nabla n_-}{\nabla n_- + \nabla n_e} + 1 + \alpha\frac{\mu_-}{\mu_e}}{1 + \mu_+(1+\alpha) + \frac{\mu_-}{\mu_e}\alpha} \right] \tag{A-18}$$

Using the relation $\frac{\nabla n_-}{\nabla n_e} = \gamma\alpha$, we get

$$\Gamma_+ = -D_+ \nabla n_+ \left[\frac{\gamma(1+\alpha)\frac{1}{1+\gamma\alpha} + \frac{\mu_-}{\mu_e}(1+\alpha)\frac{\gamma\alpha}{1+\gamma\alpha} + 1 + \alpha\frac{\mu_-}{\mu_e}}{1 + \mu_+(1+\alpha) + \frac{\mu_-}{\mu_e}\alpha} \right] \tag{A-19}$$

Factoring out a $\left(1 + \frac{\mu_-}{\mu_e}\right)/(1+\gamma\alpha)$, we have

$$\Gamma_+ = -D_+ \nabla n_+ \left[\frac{1 + \alpha\frac{\mu_-}{\mu_e}}{1 + \gamma\alpha} \, \frac{1 + \gamma + 2\alpha\gamma}{1 + \mu_+(1+\alpha)/\mu_e + \alpha\mu_-/\mu_e} \right] \tag{A-20}$$

APPENDIX B

Derivation of Ingold's equations

Ingold begins with the diffusion equations:

$$-\frac{\Gamma_e}{\mu_e} = \theta_e \frac{d}{dz} n_e + n_e E \quad \text{(B-1a)}$$

$$-\frac{\Gamma_+}{\mu_+} = \theta_g \frac{d}{dz} n_+ - n_+ E \quad \text{(B-1b)}$$

$$-\frac{\Gamma_-}{\mu_-} = \theta_g \frac{d}{dz} n_- + n_- E \quad \text{(B-1c)}$$

Together with the continuity equations:

$$\frac{d\Gamma_e}{dz} = (\nu - \lambda) n_e \quad \text{(B-2a)}$$

$$\frac{d\Gamma_+}{dz} = \nu n_e \quad \text{(B-2b)}$$

$$\frac{d\Gamma_-}{dz} = \lambda n_e \quad \text{(B-2c)}$$

and an assumption of quasineutrality,

$$n_+ = n_e + n_- \quad \text{(B-3)}$$

Ingold obtains

$$(\theta_e + \theta_g)\frac{d^2 n_e}{dz^2} + 2\theta_g \frac{d^2 n_-}{dz^2} + \left(\frac{\nu - \lambda}{\mu_e} + \frac{\nu}{\mu_+} + \frac{\lambda}{\mu_-}\right) n_e = 0 \quad \text{(B-4)}$$

Through further manipulation of equations B-1 and B-2, Ingold is able to obtain the following relation:

$$\frac{n_-'}{n_e'} = \frac{\theta_e}{\theta_g} \frac{\left(1+\frac{\theta_g}{\theta_e}\right)\frac{\lambda}{\mu_-}n_e + \left(\frac{\nu}{\mu_+} + \frac{\lambda}{\mu_-} - \frac{\theta_g}{\theta_e}\frac{\nu-\lambda}{\mu_e}\right)n_-}{\left(\frac{\nu}{\mu_+} - \frac{\lambda}{\mu_-} + \frac{\nu-\lambda}{\mu_e}\right)n_e + 2\frac{\nu-\lambda}{\mu_e}n_-} \tag{B-5}$$

Ingold fails to give any details as to how this last equation was obtained however, Dr. Alan Garscadden and Lt. Col. William Bailey offered some insight by developing a similar relation. The derivation is as follows.

Rearranging B-1b we have

$$n_+E = (n_e + n_-)E = \frac{\Gamma_+}{\mu_+} + \theta_g n_+' \tag{B-6}$$

Rearranging and adding B-1a and B-1b and replacing the electric field term with B-6 we have

$$\frac{\Gamma_+}{\mu_+} + \frac{\Gamma_e}{\mu_e} + \frac{\Gamma_-}{\mu_-} + \theta_g(n_+' + n_-') + \theta_g n_e' = 0 \tag{B-7}$$

Differentiating this last equation and substituting into B-2 gives us

$$\left(\frac{\nu-\lambda}{\mu_e} + \frac{\nu}{\mu_+} + \frac{\lambda}{\mu_-}\right)n_e + \theta_g(n_+'' + n_-'') + \theta_g n_e'' \tag{B-8}$$

Assuming $n_+'' = n_e'' + n_-''$ we get

$$(\theta_e + \theta_g)n_e'' + 2\theta_g n_-'' + \left(\frac{\nu-\lambda}{\mu_e} + \frac{\nu}{\mu_+} + \frac{\lambda}{\mu_-}\right)n_e = 0 \tag{B-9}$$

which verifies B-4.

Differentiating B-2b and making the proper substitutions we have

$$\theta_g(n_e'' + n_-'') + \frac{\nu n_e}{\mu_+} = E(n_e' + n_-') \tag{B-10}$$

Differentiating B-la and B-lc and adding both to B-10 we obtain

$$E\left(2n_-' + n_e'\left(1 + \frac{\theta_g}{\theta_e}\right)\right) = -\frac{\theta_g}{\theta_e}\left(\frac{(\nu-\lambda)n_e}{\mu_e}\right) - \frac{\lambda n_e}{\mu_-} + \frac{\nu n_e}{\mu_+} \tag{B-11}$$

Rearranging, we have

$$E = \frac{n_e\left(\frac{\nu}{\mu_+} - \frac{\lambda}{\mu_-} - \frac{\theta_g}{\theta_e}\frac{\nu-\lambda}{\mu_e}\right)}{2n_-' + n_e'(1 + \theta_g/\theta_e)} \tag{B-12}$$

Differentiating B-lc and substituting in B-12 we obtain

$$\theta_g n_-'' = \frac{-\frac{\lambda n_e}{\mu_-}\left(2n_-' + n_e'\left(1 + \frac{\theta_g}{\theta_e}\right)\right) - n_-' n_e\left(\frac{\nu}{\mu_+} - \frac{\lambda}{\mu_-} - \frac{\theta_g}{\theta_e}\frac{\nu-\lambda}{\mu_e}\right)}{2n_-' + n_e'(1 + \theta_g/\theta_e)} \tag{B-13}$$

or

$$\theta_g n_-'' = \frac{-\frac{\lambda n_e n_e'}{\mu_-}\left(1 + \frac{\theta_g}{\theta_e}\right) - n_e n_-'\left(\frac{\nu}{\mu_+} + \frac{\lambda}{\mu_-} - \frac{\theta_g}{\theta_e}\frac{\nu-\lambda}{\mu_e}\right)}{2n_-' + n_e'(1 + \theta_g/\theta_e)} \tag{B-14}$$

Using the same steps on B-la we find

$$\theta_e n_e'' = \frac{\theta_g}{\theta_e}\left\{\frac{-\frac{\nu-\lambda}{\mu_e}\left(2n_-' + n_e'(1 + \theta_g/\theta_e)\right) - n_e'\left(\frac{\nu}{\mu_+} - \frac{\lambda}{\mu_-} - \frac{\theta_g}{\theta_e}\frac{\nu-\lambda}{\mu_e}\right)}{2n_-' + n_e'(1 + \theta_g/\theta_e)}\right\} n_e$$

$$= \frac{\theta_g}{\theta_e}\left\{\frac{\frac{(\nu-\lambda)2n_-'}{\mu_e} + \left(\frac{\nu}{\mu_+} - \frac{\lambda}{\mu_-} + \frac{\nu-\lambda}{\mu_e}\right)n_e'}{2n_-' + n_e'(1 + \theta_g/\theta_e)}\right\} n_e \tag{B-15}$$

Dividing B-14 by B-15 we obtain

$$\frac{n_-''}{n_e''}=\frac{\frac{\lambda}{\mu_-}\left(1+\frac{\theta_g}{\theta_e}\right)n_e'+\left(\frac{\nu}{\mu_+}+\frac{\lambda}{\mu_-}-\frac{\theta_g}{\theta_e}\frac{\nu-\lambda}{\mu_e}\right)n_-'}{\frac{\theta_g}{\theta_e}\left\{\frac{(\nu-\lambda)}{\mu_e}2n_-'+\left(\frac{\nu}{\mu_+}-\frac{\lambda}{\mu_-}+\frac{\nu-\lambda}{\mu_e}\right)n_e'\right\}} \tag{B-16}$$

This differs from Ingold in that it relates the ratio of second derivatives to first derivatives whereas Ingold relates the ratio of first derivatives to undifferentiated terms. It can be shown that B-16 reduces to B-5 if the ratio can be written as two separate equations, i.e.

$$n_-''=\frac{\lambda}{\mu_-}\left(1+\frac{\theta_g}{\theta_e}\right)n_e'+\left(\frac{\nu}{\mu_+}+\frac{\lambda}{\mu_-}-\frac{\theta_g}{\theta_e}\frac{\nu-\lambda}{\mu_e}\right)n_-'$$

$$n_e''=\frac{\theta_g}{\theta_e}\left\{\frac{(\nu-\lambda)}{\mu_e}2n_-'+\left(\frac{\nu}{\mu_+}-\frac{\lambda}{\mu_-}+\frac{\nu-\lambda}{\mu_e}\right)n_e'\right\}$$

but, as was shown in the derivation of B-16, this can not be done since the term $n_e/(2n_-'+n_e'(1+\theta_g/\theta_e))$ common to both, is cancelled out when the ratio is taken. Thus, B-5 cannot be properly established and, since it is an essential relation to Ingold's development (as will be shown later), his final results must be viewed with skepticism.

Let us now begin the task of tracing out Ingold's development. It should be noted that only those equations specifically referred to as Ingold's were obtained from Ingold's notes. All other equations are conjecture as to how Ingold's equations were arrived at.

We begin by integrating B-2 with respect to z.

$$\Gamma_e = (\nu - \lambda) N_e \tag{B-17a}$$

$$\Gamma_+ = \nu N_e \tag{B-17b}$$

$$\Gamma_- = \lambda N_e \tag{b-17c}$$

Thus, B-1a can be written as

$$\begin{aligned} E &= -\theta_e \frac{n_e'}{n_e} - \frac{\Gamma_e}{\mu_e n_e} \frac{\Gamma_+}{\Gamma_+} \\ &= -\theta_e \frac{n_e'}{n_e} - \frac{\Gamma_+}{\nu N_e} \frac{(\nu-\lambda) N_e}{\mu_e n_e} \\ &= -\theta_e \frac{x'}{x} - \frac{\nu-\lambda}{\nu} \frac{\Gamma_+}{\mu_e n_{eo} x} \end{aligned} \tag{B-18}$$

Similarly, B-1c can be written as

$$E = -\theta_g \frac{y'}{y} - \frac{\lambda}{\nu} \frac{\Gamma_+}{\mu_- n_{-o} y} \tag{B-19}$$

Equating B-18 to B-19, we obtain Ingold's relation

$$\theta_e \frac{x'}{x} + \frac{\nu-\lambda}{\nu} \frac{\Gamma_+}{\mu_e n_{eo} x} = \theta_g \frac{y'}{y} + \frac{\lambda}{\nu} \frac{\Gamma_+}{\mu_- n_{-o} y} \tag{B-20}$$

Substituting B-17 into B-7, we have

$$\frac{\Gamma_+}{\mu_+} + \frac{\Gamma_+}{\nu N_e} \frac{(\nu-\lambda) N_e}{\mu_e} + \frac{\Gamma_+ \lambda N_e}{\nu N_e \mu_-} + \theta_g n_+' + \theta_g n_-' + \theta_e n_e' = 0 \tag{B-21}$$

Solving for Γ_+, we obtain

$$\Gamma_+ = \frac{-(\theta_e + \theta_g) n_{eo} x' + 2\theta_g n_{-o} y'}{\frac{\nu-\lambda}{\nu} \frac{1}{\mu_e} + \frac{\lambda}{\nu} \frac{1}{\mu_-} + \frac{1}{\mu_+}} \tag{B-22}$$

Substituting B-22 into B-20, we get Ingold's next relation

$$y\left\{\theta_e x' - \frac{\nu-\lambda}{\mu_e}\left[\frac{(\theta_e+\theta_g)x' + 2\theta_g \frac{n_{-o}}{n_{eo}} y'}{\frac{\nu-\lambda}{\mu_e} + \frac{\nu}{\mu_+} + \frac{\lambda}{\mu_-}}\right]\right\} = x\left\{\theta_g y' - \frac{\lambda}{\mu_-}\left[\frac{(\theta_e+\theta_g)\frac{n_{eo}}{n_{-o}} x' + 2\theta_g y'}{\frac{\nu-\lambda}{\mu_e} + \frac{\nu}{\mu_+} + \frac{\lambda}{\mu_-}}\right]\right\} \quad \text{(B-23)}$$

Differentiating B-23 and evaluating it at z=0 (where $x_0 = y_0 = 1$ and $x_0' = y_0' = 0$), we obtain the following Ingold equation

$$\theta_e x_0'' - \frac{\nu-\lambda}{\mu_e}\left[\frac{(\theta_e+\theta_g)x_0'' + 2\theta_g \frac{n_{-o}}{n_{eo}} y_0''}{\frac{\nu-\lambda}{\mu_e} + \frac{\nu}{\mu_+} + \frac{\lambda}{\mu_-}}\right] = \theta_g y_0'' - \frac{\lambda}{\mu_-}\left[\frac{(\theta_e+\theta_g)x_0'' + 2\theta_g \frac{n_{-o}}{n_{eo}} y_0''}{\frac{\nu-\lambda}{\mu_e} + \frac{\nu}{\mu_+} + \frac{\lambda}{\mu_-}}\right]\frac{n_{eo}}{n_{-o}} \quad \text{(B-24)}$$

Defining a,b,c and d as

$$a = \frac{\theta_e}{\theta_g}\left(1 + \frac{\theta_g}{\theta_e}\right)\frac{\lambda}{\mu_-} n_{eo} \qquad b = \left(\frac{\nu}{\mu_+} + \frac{\lambda}{\mu_-} - \frac{\theta_g}{\theta_e}\frac{\nu-\lambda}{\mu_e}\right) n_{-o}$$

$$c = \left(\frac{\nu}{\mu_+} - \frac{\lambda}{\mu_-} + \frac{\nu-\lambda}{\mu_e}\right) n_{-o} \qquad d = 2\,\frac{\nu-\lambda}{\mu_e}\,\frac{n_{-o}^2}{n_{eo}} \quad \text{(B-25)}$$

B-5 can be written as

$$x'(ax + by) = y'(cx + dy) \quad \text{(B-26)}$$

Differentiating B-25 and evaluating it at z=0 we have

$$x_0''(a+b) = y_0''(c+d) \quad \text{(B-27)}$$

It will be shown later that

$$\frac{a+b}{c+d} = \frac{\theta_e}{\theta_g} \tag{B-28}$$

Thus, we obtain Ingold's relation

$$X_o'' \frac{\theta_e}{\theta_g} = Y_o'' \tag{B-29}$$

But, as was pointed out earlier, B-5 can not be properly established. Instead, B-16 must be used which yields

$$X''(ax'+by') = y''(cx'+dy') \tag{B-30}$$

instead of B-5. Since B-29 is needed in the next step of Ingold's development and B-26 in derivations following that, all results past this point must be considered dubious at best.

Assuming B-29 is true, and substituting it into B-24 and simplifying, we obtain one of Ingold's key results

$$\frac{\nu-\lambda}{\mu_e} = \frac{\lambda}{\mu_-} \frac{n_{eo}}{n_{-o}} \tag{B-31}$$

Let us now establish the following two equations given by Ingold.

$$a+b = \frac{\theta_e}{\theta_g}\left[1+\left(1+\frac{n_{-o}}{n_{eo}}\right)\frac{\mu_+/\mu_-}{\frac{\mu_e}{\mu_-}+\frac{n_{-o}}{n_{eo}}}\right] \tag{B-32}$$

$$c+d = 1 + \left(1 + \frac{n_{-o}}{n_{eo}}\right) \frac{\mu_+/\mu_-}{\frac{\mu_e}{\mu_-} + \frac{n_{-o}}{n_{eo}}} \qquad \text{(B-33)}$$

From B-25 and B-31, we have

$$a+b = \frac{\theta_e}{\theta_g}\left[\frac{\lambda}{\mu_-} n_{eo} + \frac{\theta_g}{\theta_e}\frac{\lambda}{\mu_-} n_{eo} + \left(\frac{\lambda \mu_e n_{eo}}{\mu_+ \mu_- n_{-o}}\right) n_{-o} + \frac{\lambda}{\mu_-} n_{-o} - \frac{\theta_g}{\theta_e}\frac{\lambda n_{eo}}{\mu_- n_{-o}} n_{-o}\right] \qquad \text{(B-34)}$$

We now factor out an λ which will cancel an λ that is factored out of the expression c+d, i.e. we essentially redefine a,b,c and d such that

$$\frac{ax+by}{cx+dy} = \frac{\frac{1}{\lambda}(ax+by)}{\frac{1}{\lambda}(cx+dy)} \qquad \text{(B-35)}$$

Therefore, B-34 reduces to

$$a+b = \frac{\theta_e}{\theta_g}\left(\frac{n_{eo}}{\mu_-} + \frac{\mu_e n_{eo}}{\mu_+ \mu_-} + \frac{n_{-o}}{\mu_+} + \frac{n_{-o}}{\mu_-}\right) \qquad \text{(B-36)}$$

Placing all terms over a common denominator of $\mu_+ \mu_-$ and factoring out a $\frac{\mu_e n_{eo} + \mu_- n_{-o}}{\mu_+ \mu_-}$, we have

$$a+b = \frac{\theta_g}{\theta_e}\left(\frac{\mu_e n_{eo} + n_{-o}\mu_-}{\mu_+ \mu_-}\right)\left(\frac{\mu_+ n_{eo} + \mu_e n_{eo} + \mu_- n_{-o} + \mu_+ n_{-o}}{\mu_e n_{eo} + \mu_- n_{-o}}\right) \qquad \text{(B-37)}$$

Also, from B-25, B-31 and B-35, we have

$$c+d = \frac{\mu_e n_{eo}}{\mu_+ \mu_-} + \frac{n_{eo}}{\mu_-} + \frac{n_{-o}}{\mu_-} + \frac{n_{-o}}{\mu_+} \tag{B-38}$$

Again factoring out $(\mu_e n_{eo} + n_{-o}\mu_-)/\mu_+\mu_-$ we obtain

$$c+d = \left(\frac{\mu_e n_{eo}}{\mu_+ \mu_-}\right)\left(\frac{\mu_+ n_{eo} + \mu_+ n_{-o} + \mu_e n_{eo} + \mu_- n_{-o}}{\mu_e n_{eo} + \mu_- n_{-o}}\right) \tag{B-39}$$

Again we redefine a,b,c and d such that the terms $(\mu_e n_{eo} + n_{-o}\mu_-)/\mu_+\mu_-$ cancel in equations B-37 and B-38.

We will now show that

$$1 + \left(1 + \frac{n_{-o}}{n_{eo}}\right)\frac{\mu_+/\mu_-}{\frac{\mu_e}{\mu_-} + \frac{n_{-o}}{n_{eo}}} = \frac{\mu_+ n_{eo} + \mu_e n_{eo} + \mu_+ n_{-o} + \mu_- n_{-o}}{\mu_e n_{eo} + \mu_- n_{-o}} \tag{B-40}$$

thus proving that B-37 and B-39 reduce to B-32 and B-33.

Since

$$\frac{\mu_+/\mu_-}{\frac{\mu_e}{\mu_-} + \frac{n_{-o}}{n_{eo}}} = \frac{\mu_+/\mu_-}{\frac{\mu_e n_{eo}}{\mu_- n_{eo}} + \frac{\mu_- n_{-o}}{\mu_- n_{eo}}}$$

$$= \frac{\mu_+/\mu_-}{\frac{\mu_e n_{eo} + \mu_- n_{-o}}{\mu_- n_{eo}}} = \frac{n_{eo}\mu_+}{\mu_e n_{eo} + \mu_- n_{-o}} \tag{B-41}$$

we can write

$$\left(1 + \frac{n_{-o}}{n_{eo}}\right)\frac{\mu_+/\mu_-}{\frac{\mu_e}{\mu_-} + \frac{n_{-o}}{n_{eo}}} = \frac{\mu_+ n_{eo} + \mu_+ n_{-o}}{\mu_e n_{eo} + \mu_- n_{-o}} \tag{B-42}$$

Therefore

$$1+\left(1+\frac{n_{-o}}{n_{eo}}\right)\frac{\mu_+/\mu_-}{\frac{\mu_e}{\mu_-}+\frac{n_{-o}}{n_{eo}}}=\frac{\mu_e n_{eo}+\mu_- n_{-o}+\mu_+ n_{eo}+\mu_+ n_{-o}}{\mu_e n_{eo}+\mu_+ n_{-o}} \qquad \text{(B-43)}$$

which verifies B-40. Although keeping the terms we cancelled out by redefining a,b,c and d would have given expressions different than B-31 and B-32, the important relation needed for Ingold's development, equation B-28, would still be valid.

Continuing Ingold's development, Ingold next assumes that

$$y=u+fx \qquad \text{(B-44)}$$

where u is some function of the spatial coordinate and f is a constant. It follows then that

$$\frac{dy}{dx}=\frac{du}{dx}+f=\frac{ax+by}{cx+dy}=\frac{(a+bf)x+bu}{(c+df)x+du} \qquad \text{(B-45)}$$

or

$$\frac{du}{dx}=\frac{\left[(a+bf)-f(c+df)\right]x+(b-df)u}{(c+df)x+du} \qquad \text{(B-46)}$$

Ingold then makes the seemingly arbitrary assumption

$$(a+bf)-f(c+df)=0 \tag{B-47}$$

which fixes the value of f at

$$f=\frac{b-c\pm\sqrt{(b-c)^2+4ad}}{2d} \tag{B-48}$$

This allows Ingold to write B-46 as

$$\frac{du}{dx}=\frac{mu}{nx+pu} \tag{B-49}$$

where m=b-df, n=c+df and p=d. The solution to B-49 is

$$x=Au^{\frac{n}{m}}+\frac{p}{m-n}u \tag{B-50}$$

Since y=x=1 at z=0, we can see from B-44 that u=1-f at z=0. Using these values for x,y and u determines A, which yields Ingold's

$$A=\frac{1-\frac{p}{m-n}(1-f)}{(1-f)^{n/m}} \tag{B-51}$$

Substituting B-50 into B-44 we obtain Ingold's expression for y

$$y = Cu^s + Du \tag{B-52}$$

where

$$C = Af \; ; \; D = Bf + 1 \; ; \; B = \frac{p}{m-n} \text{ and } s = \frac{n}{m} \tag{B-53}$$

From B-9, using the variables x and y, we have

$$x'' + \frac{2h}{\gamma+1} y'' + k^2 x = 0 \tag{B-54}$$

where

$$k^2 = \frac{\frac{\nu-\lambda}{\mu_e} + \frac{\nu}{\mu_+} + \frac{\lambda}{\mu_-}}{\theta_e + \theta_g} \tag{B-55}$$

Thus, taking the second derivatives of B-50 and B-52 and substituting into B-54, we obtain Ingold's key result:

$$\left[\left(A + \frac{2h}{\gamma+1} C\right) s u^{s-1} + B + \frac{2h}{\gamma+1} D\right] u'' + \left(A + \frac{2h}{\gamma+1} C\right) s(s-1) u^{s-2} u'^2 + k^2 (Au^s + Bu) \tag{B-56}$$

APPENDIX C

Derivation of Lee's Recursion Relations

Lee's four first order differential equations including boundary conditions are (7:4701)

$$z^{-1}(z y_3)' = [(\nu - \lambda)/\lambda]\, y_1 + (\varphi/\lambda)\, y_2 \tag{C-1a}$$

$$z^{-1}(z y_4)' = y_1 - (\varphi/\lambda)\, y_2 \tag{C-1b}$$

$$y_1' + 2\sigma y_2' = -A y_3 - B y_4 \tag{C-1c}$$

$$y_1' y_2 - \sigma y_2' y_1 = (B - A) y_1 y_4 - C y_2 y_3 \tag{C-1d}$$

and

$$y_1(0) = 1;\quad y_3(0) = 0;\quad y_4(0) = 0;\quad y_1(1) = 0;\quad y_2(1) = 0 \tag{C-2}$$

where

$$y_1 \equiv \frac{n_e(Rz)}{n_{eo}};\qquad y_2 \equiv \frac{n_-(Rz)}{n_{eo}}$$

$$y_3 \equiv \frac{j_e(Rz)}{\alpha n_{eo} R}\qquad y_4 \equiv \frac{j_-(Rz)}{\alpha n_{eo} R} \tag{C-3}$$

and

$$\sigma \equiv \theta_g/\theta_e;\qquad A \equiv \frac{m_+ \nu_+ \lambda R^2}{\theta_e}$$

$$B \equiv A + \frac{m_- \nu_- \lambda R^2}{\theta_e};\qquad C \equiv \frac{m_e \nu_e \lambda R^2}{\theta_e} \tag{C-4}$$

Also, the derivatives are in terms of the normalized coordinate, z, and j represents the charged species flux density. The momentum transfer frequencies are given by Lee as (7:4703)

$$m_- \nu_- = 4.8693 \times 10^{-16}\ \text{g sec}^{-1}\ \text{torr}^{-1} \times P$$

$$m_+ \nu_+ = 8.8508 \times 10^{-16}\ \text{g sec}^{-1}\ \text{torr}^{-1} \times P$$

$$m_e \nu_e = 2.9127 \times 10^{-18}\ \text{g sec}^{-1}\ \text{torr}^{-1} \times P \tag{C-5}$$

Since Lee uses cylindrical coordinates, a singularity is encountered at z=0. To get around this, Lee uses power series approximations near zero given by (7:4703)

$$y_1 = p_0 + \sum_1^{\infty} p_{2k} z^{2k} \, ; \quad y_2 = q_0 + \sum_1^{\infty} q_{2k} z^{2k}$$
$$y_3 = \sum_1^{\infty} r_{2k-1} z^{2k-1} \, ; \quad y_4 = \sum_1^{\infty} s_{2k-1} z^{2k-1} \tag{C-6}$$

where

$$p_0 = 1 \quad \text{and} \quad q_0 = h \tag{C-7}$$

Substituting C-3 into C-1a, we have

$$\frac{1}{z}\left(\sum_1^{\infty} r_{2k-1} z^{2k-1} + z\left(\sum_1^{\infty} r_{2k-1}(2k-1) z^{2k-2}\right)\right) = \frac{\nu-\lambda}{\lambda}\left(p_0 + \sum_1^{\infty} p_{2k} z^{2k}\right) + \frac{\varphi}{\lambda}\left(q_0 + \sum_1^{\infty} q_{2k} z^{2k}\right) \tag{C-8}$$

Redefining the dummy variable of summation on the right side as k=k-1, we obtain

$$\sum_1^{\infty} r_{2k-1} z^{2k-2} + \sum_1^{\infty} r_{2k-1}(2k-1) z^{2k-2} = \frac{\nu-\lambda}{\lambda}\left(p_0 + \sum_2^{\infty} p_{2k-2} z^{2k-2}\right) + \frac{\varphi}{\lambda}\left(q_0 + \sum_2^{\infty} q_{2k-2} z^{2k-2}\right) \tag{C-9}$$

If we include p_0 and q_0 in the summation on the right side of C-9, we have

$$\sum_1^{\infty} r_{2k-1} z^{2k-2} + \sum_1^{\infty} r_{2k-1}(2k-1) z^{2k-2} = \frac{\nu-\lambda}{\lambda}\left(\sum_1^{\infty} p_{2k-2} z^{2k-2}\right) + \frac{\varphi}{\lambda}\left(\sum_1^{\infty} q_{2k-2} z^{2k-2}\right) \tag{C-10}$$

Again redefining the dummy variable $k = \ell + 1$ and equating like powers of z, we obtain Lee's eq. 4.5 (7:4703)

$$r_{2\ell+1} = \frac{1}{2(\ell+1)}\left(\frac{v-\lambda}{\lambda} p_{2\ell} + \frac{\varphi}{\lambda} q_{2\ell}\right) \quad \ell = 0, 1, 2 \ldots \tag{C-11}$$

Substituting C-3 into C-1b, we have

$$\frac{1}{z}\left(\sum_1^\infty s_{2k-1} z^{2k-1} + z\left(\sum_1^\infty s_{2k-1}(2k-1) z^{2k-2}\right)\right) = \sum_0^\infty p_{2k} z^{2k} - \frac{\varphi}{\lambda}\sum_0^\infty q_{2k} z^{2k} \tag{C-12}$$

where p_0 and q_0 have again been included in the summation of y_1 and y_2. We redefine $k = k+1$ on the left side of C-12 to obtain

$$\sum_1^\infty s_{2k+1} z^{2k} + \sum_1^\infty s_{2k+1}(2k+1) z^{2k} = \sum_1^\infty p_{2k} z^{2k} - \frac{\varphi}{\lambda}\sum_1^\infty q_{2k} z^{2k} \tag{C-13}$$

Equating like powers of z, we obtain Lee's eq. 4.6 (7:4703)

$$s_{2k+1} = \frac{1}{2(k+1)}\left(p_{2k} - \frac{\varphi}{\lambda} q_{2k}\right) \quad k = 0, 1, 2 \ldots \tag{C-14}$$

Substituting C-3 into C-1c, we have

$$\sum_1^\infty p_{2k}(2k) z^{2k-1} + 2\sigma \sum_1^\infty q_{2k} 2k\, z^{2k-1} = -A \sum_1^\infty r_{2k-1} z^{2k-1} - B \sum_1^\infty s_{2k-1} z^{2k-1} \tag{C-15}$$

Equating like powers of z, we obtain

$$p_{2k} + 2\sigma q_{2k} = -\frac{1}{2k}\left(A r_{2k-1} + B s_{2k-1}\right) \tag{C-16}$$

which is identical to Lee's eq. 4.7 (7:4703).

Finally, substituting C-3 into C-1d, we have

$$\left(\sum_1^\infty p_{2k}(2k)z^{2k-1}\right)\left(q_0 + \sum_1^\infty q_{2k}z^{2k}\right) - \sigma\left(\sum_1^\infty q_{2k}(2k)z^{2k-1}\right)\left(\sum_1^\infty p_{2k}z^{2k} + p_0\right)$$

$$= (B-A)\left(p_0 + \sum_1^\infty p_{2k}z^{2k}\right)\left(\sum_1^\infty s_{2k-1}z^{2k-1}\right) - C\left(q_0 + \sum_1^\infty q_{2k}z^{2k}\right)\left(\sum_1^\infty r_{2k-1}z^{2k-1}\right) \tag{C-17}$$

Multiplying the first two terms together, we obtain

$$q_0 \sum_1^\infty p_{2k}(2k)z^{2k-1} + \sum_1^\infty p_{2k}(2k)z^{2k-1}\sum_1^\infty q_{2\ell}z^{2\ell} \tag{C-18}$$

where we have changed the summation variable from k to ℓ in the second term for clarification. Simplifying the second term in C-16, we have

$$\sum_{k=1}^{n} p_{2k}(2k)\left[\sum_{m-k=1}^{n} q_{2(m-k)}z^{2m-1}\right] \quad \text{as } n \to \infty \tag{C-19}$$

where $m = k + \ell$. Since we are interested in equating like terms of z, we want all combinations of $\ell + k$ that yield the same m. All such combinations can be expressed as

$$\sum_{k=1}^{m-1} p_{2k}(2k)\, q_{2(m-k)}\, z^{2m-1} \tag{C-20}$$

for any given m. Likewise,

$$-\sigma\left(\sum_{1}^{\infty} q_{2k}(2k)z^{2k-1}\right)\left(\sum_{1}^{\infty} p_{2k} z^{2k}\right) \quad \text{becomes} \quad -\sigma\sum_{k=1}^{m-1} q_{2k}(2k)\,p_{2(m-k)}z^{2m-1}$$

$$-C\left(\sum_{1}^{\infty} r_{2k-1} z^{2k-1}\right)\left(\sum_{1}^{\infty} q_{2k} z^{2k}\right) \quad \text{becomes} \quad -C\sum_{k=1}^{m-1} r_{2k-1}\,q_{2(m-k)}z^{2m-1} \qquad \text{(C-21)}$$

Therefore, upon substituting m for k in the remaining sums and equating like coefficients of z, we obtain

$$q_0 p_{2m}(2m)+\sum_{k=1}^{m-1} p_{2k}(2k)q_{2(m-k)}-\sigma p_0 q_{2m}(2m)-\sigma\sum_{k=1}^{m-1} q_{2k}(2k)p_{2(m-k)}$$
$$=(B-A)p_0 S_{2m-1}+(B-A)\sum_{k=1}^{m-1} S_{2k-1}p_{2(m-k)}-Cq_0 r_{2m-1}-C\sum_{k=1}^{m-1} r_{2k-1}q_{2(m-k)} \qquad \text{(C-22)}$$

Since $q_0=h$ and $p_0=1$ we have, after changing summation variables from m to n and from k to l and simplifying,

$$h p_{2n}-\sigma q_{2n}=\frac{1}{2n}\Big[(B-A)\Big(S_{2n-1}+\sum_{\ell=1}^{n-1}S_{2\ell-1}p_{2(n-\ell)}\Big)+\sigma\sum_{\ell=1}^{n-1}2\ell q_{2\ell}p_{2(n-\ell)}$$
$$-\sum_{\ell=1}^{n-1}2\ell p_{2\ell}q_{2(n-\ell)}-C\Big(h r_{2n-1}+\sum_{\ell=1}^{n-1}r_{2\ell-1}q_{2(n-\ell)}\Big)\Big] \qquad \text{(C-23)}$$

This differs from Lee's eq. 4.8 (7:4703) in that the third term on the right side of the equation contains an extra ℓ , i.e.

$$\sum_{\ell=1}^{n-1}2\ell p_{2\ell}q_{2(n-\ell)} \quad \text{vs. Lee's} \quad \sum_{\ell=1}^{n-1}2 p_{2\ell}q_{2(n-\ell)}$$

After discussing the discrepancy with Dr. Lee, it was discovered that the missing ℓ was merely a typographical error in his paper.

The singularity at the wall is dealt with by the formal

approximation (7:4703)

$$y_1 = a_1(1-z) + O(1-z)^2 \tag{C-24a}$$
$$y_2 = b_1(1-z) + O(1-z)^2 \tag{C-24b}$$
$$y_3 = c_0 + c_1(1-z) + O(1-z)^2 \tag{C-24c}$$
$$y_4 = d_0 + d_1(1-z) + O(1-z)^2 \tag{C-24d}$$

Lee then substitutes C-24 into C-1 to obtain (7:4703)

$$y_1 = a_1(1-z) \qquad y_2 = b_1(1-z) \qquad \text{(C-25)}$$
$$y_3 = c_0(2-z) \qquad y_4 = d_0(2-z)$$

with

$$c_0 = D^{-1}[a_1(B-A)(a_1+2\sigma b_1) + a_1 b_1 B(1-\sigma)] \tag{C-26a}$$
$$d_0 = -D^{-1}[A a_1 b_1(1-\sigma) - C b_1(a_1+2\sigma b_1)] \tag{C-26b}$$
$$D = a_1 A(B-A) + BCb_1 \tag{C-26c}$$

where terms of order $(1-z)^2$ are neglected. These equations will now be derived for verification.

Substituting C-24 into C-1c gives

$$-a_1 - 2\sigma b_1 = -A(c_0 + c_1(1-z)) - B(d_0 + d_1(1-z)) \tag{C-27}$$

simplifying, we have

$$d_0 + d_1(1-z) = \left[a_1 + 2\sigma b_1 - A(c_0 + c_1(1-z))\right] / B \tag{C-28}$$

Now substituting C-24 into C-1d, we obtain

$$-a_1 b_1 (1-z) + \sigma b_1 a_1 (1-z) = (B-A) a_1 (1-z)\left[d_0 + d_1(1-z)\right]$$
$$- c\, b_1 (1-z)\left[c_0 + c_1(1-z)\right] \tag{C-29}$$

Substituting C-28 into C-29, we have

$$-a_1 b_1 (1-z) + \sigma b_1 a_1 (1-z) = (B-A) a_1 (1-z)\left[a_1 + 2\sigma b_1 - A(c_0 + c_1(1-z))\right]/B$$
$$- c\, b_1 (1-z)\left[c_0 + c_1(1-z)\right] \tag{C-30}$$

Neglecting terms of order $(1-z)^2$ and solving for c_0, we obtain

$$c_0 = \frac{a_1 (B-A)(a_1 + 2\sigma b_1) + a_1 b_1 B(1-\sigma)}{a_1 A (B-A) + B c b_1} \tag{C-31}$$

Thus verifying C-26a. Now solving C-29 for $c_0 + c_1(1-z)$ and substituting it into C-28, we get

$$d_0 + d_1(1-z) = \frac{a_1 + 2\sigma b_1}{B} - \frac{A\left[(B-A) a_1 (1-z)\left[d_0 + d_1(1-x)\right] + a_1 b_1 (1-z) - \sigma b_1 a_1 (1-z)\right]}{B c b_1 (1-z)} \tag{C-32}$$

Multiplying through by $BCb_1(1-z)$, neglecting terms of order $(1-z)^2$ and solving for d_0 , we obtain

$$d_0 = \frac{Cb_1(a_1+2\sigma b_1) - Aa_1b_1(1-\sigma)}{a_1A(B-A) + BCb_1} \tag{C-33}$$

Thus verifying C-26b.

We must now attempt to verify C-25. We see that the expressions for y_1 and y_2 in C-25 are the same as C-24 but those for y_3 and y_4 change. Let's begin with C-1b.

$$zy_4' + y_4 = z\left(y_1 - \frac{\varphi}{\lambda} y_2\right) \tag{C-34}$$

Adding $-y_4'$ and $-y_1 + \frac{\varphi}{\lambda} y_2$ to both sides, we can write

$$-y_4(1-z) + y_4 - y_1 + \frac{\varphi}{\lambda} y_2 = \left(-y_2 + \frac{\varphi}{\lambda} y_2\right)(1-z) - y_4' \tag{C-35}$$

Substituting C-24 into C-35 and ignoring terms of order $(1-z)^2$, we obtain

$$d_1(1-z) + d_0 + d_1(1-z) - a_1(1-z) + \frac{\varphi}{\lambda} b_1(1-z) = d_1 \tag{C-36}$$

or

$$\left(2d_1 - a_1 + \frac{\varphi}{\lambda} b_1\right)(1-z) = d_1 - d_0 \tag{C-37}$$

Since the left side depends on $(1-z)$ and the right side does not, we must require

$$d_1 = d_0 \qquad \text{(C-38)}$$

It is important to note, at this point, that the sum of the coefficients on the left side cannot be equated to zero since, strictly speaking, equations C-24 should be expressed as an infinite series in powers of $(1-z)$. Thus, the derivation of y_4 would be

$$y_4' = -d_1 - 2d_2(1-z) + \text{higher order terms} \qquad \text{(C-39)}$$

where d_2 would have to be included on the left side of C-37. (Credit for the above information is attributed to Dr. Lee for his informative discussion on the apparent contradictions created by ignoring second order terms.)

Looking now at C-1a, we have

$$z y_3' + y_3 = \left\{\left[(\nu-\lambda)/\lambda\right] y_1 + \frac{\varphi}{\lambda} y_2\right\} z \qquad \text{(C-40)}$$

Adding $-y_3'$ and $-\left\{\left[(\nu-\lambda)/\lambda\right]y_1+\frac{\varphi}{\lambda}y_2\right\}$ to both sides, we obtain

$$-y_3'(1-z) + y_3 - \left\{\left[(\nu-\lambda)/\lambda\right] y_1 + \frac{\varphi}{\lambda} y_2\right\} =$$

$$-(1-z)\left\{\left[(\nu-\lambda)/\lambda\right] y_1 + \frac{\varphi}{\lambda} y_2\right\} - y_3' \qquad \text{(C-41)}$$

Substituting C-24 into C-41 and ignoring terms of order $(1-z)^2$, we obtain, after rearranging

$$\left[c_1 - ((\nu-\lambda)/\lambda)a_1 - \frac{\phi}{\lambda} b_1\right](1-z) = c_1 - c_0 \tag{C-42}$$

Again, equating zeroth order terms in $(1-z)$, we have

$$c_1 = c_0 \tag{C-43}$$

Using C-38 and C-43, we see that C-24c and C-24d reduce to

$$y_3 = c_0(2-z) \qquad y_4 = d_0(2-z) \tag{C-44}$$

thus verifying C-25.

APPENDIX D

Explanation of Lee's Code

List of Variables

A,B,C	-constants used in Lee's empirical formula for calculating ionization rate
A1-4	-constants used in calculating polynomial approximation for erfc($C_i/\sqrt{T_e}$)
AA,BB,CC	-A,B and C as given in eqs. C-4
AA1,BB1, CC1,DD1, DD	- a, b, c, d and D as referred to in C-24,C-25 and C-26
ALPHA	-dissociative attachment rate (sec^{-1})
BETA	-ionization rate (sec^{-1})
C1-11	-constants used in calculating Lee's empirical formula for dissociative attachment rate
DIF1	-difference between Y3# calculated using boundary condition equations and the value calculated at the boundary using the integration routine
DIF2	-difference between Y4# calculated using boundary condition equations and the value calculated at the boundary using the integration routine
DT#	-mesh interval, Δz, used in integration routine
EE	-electric field calculated from electron diffusion equation
EN	-electric field calculated from negative ion diffusion equation
EP2	-distance from wall where integration routine is forced to meet boundary condition.
EI	-constant used in Lee's empirical formula for ionization rate
FNERF	-Polynomial approximation for erfc($C_i/\sqrt{T_e}$) (1:299)
GAMMA	-associative detachment rate (sec^{-1})

H	-on axis ratio of negative ions to electrons
K1#-4#	-first derivative approximations used in integration routine
I	-do loop counter
K	-do loop counter
L%	-number of terms retained in polynomial approximations given by C-6
MNUE, MNUP,MNUN	-constants given by C-5
NK	-do loop counter
NN	-number of mesh points used in integration routine
NO2	-number of oxygen molecules per cm^{-3}
P	-pressure in torr
P#(L%)	-coefficients for polynomial approximation of Y1#
Q#(L%)	-coefficients for polynomial approximation of Y2#
R#(L%)	-coefficients for polynomial approximation of Y3#
S#(L%)	-coefficients for polynomial approximation of Y4#
SIGMA	-ratio of backround gas temperature to electron temperature
SUM1#-4#	-summations indicated in C-22 and C-23
T0	-backround gas temperature (ev)
T1	-initial value of z where integration begins
TE	-electron temperature (ev)
TN	-value of z where integration ends. EP2=1-TN
Y1#-4#	- y_1, y_2, y_3 and y_4 as indicated by C-3
Y1D#-4D#	-derivatives of Y1#-4# in power series approximations
Y10#-40#	-values of Y1#-4# from previous mesh point

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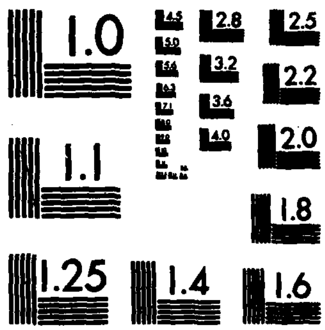

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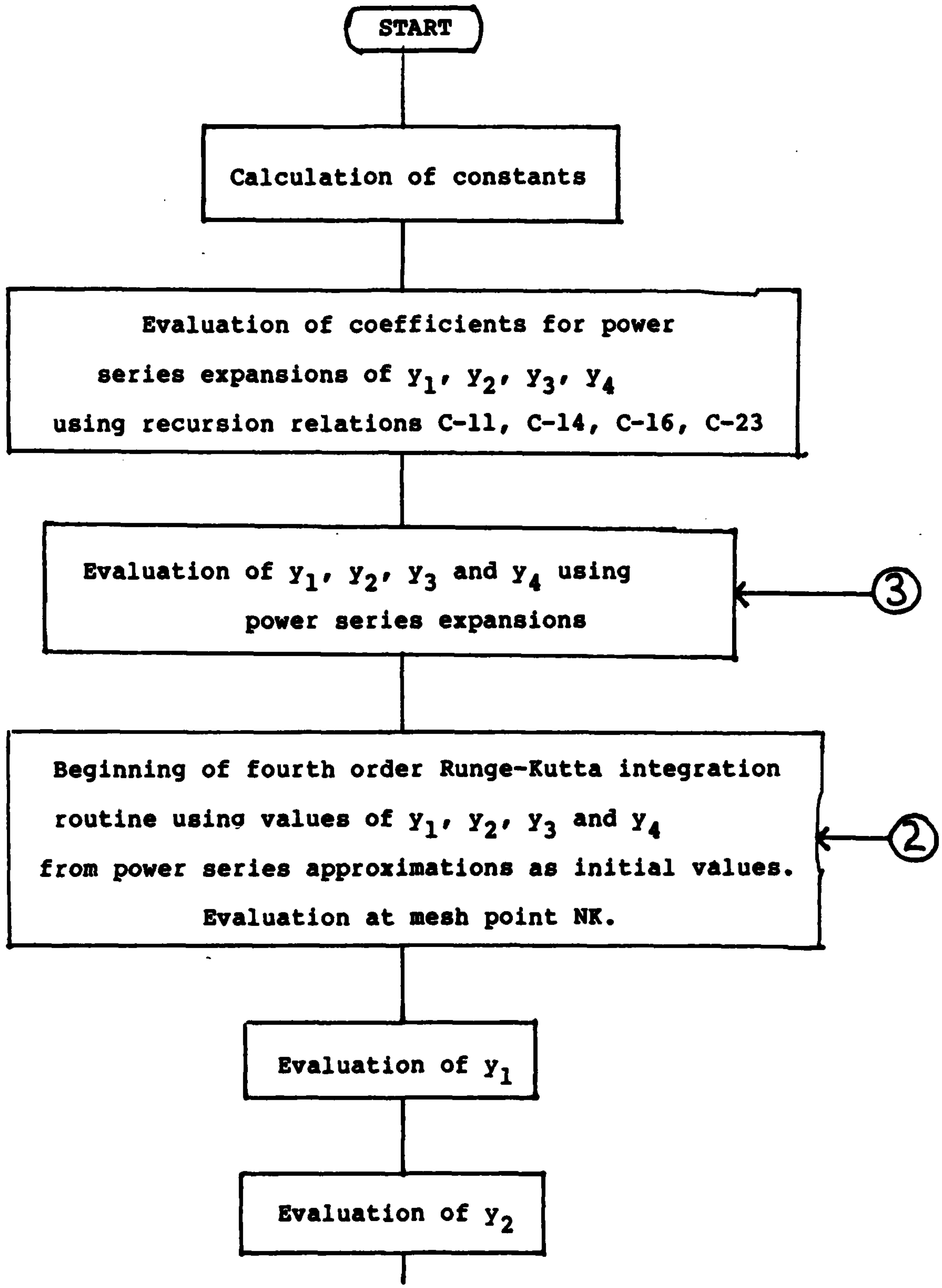
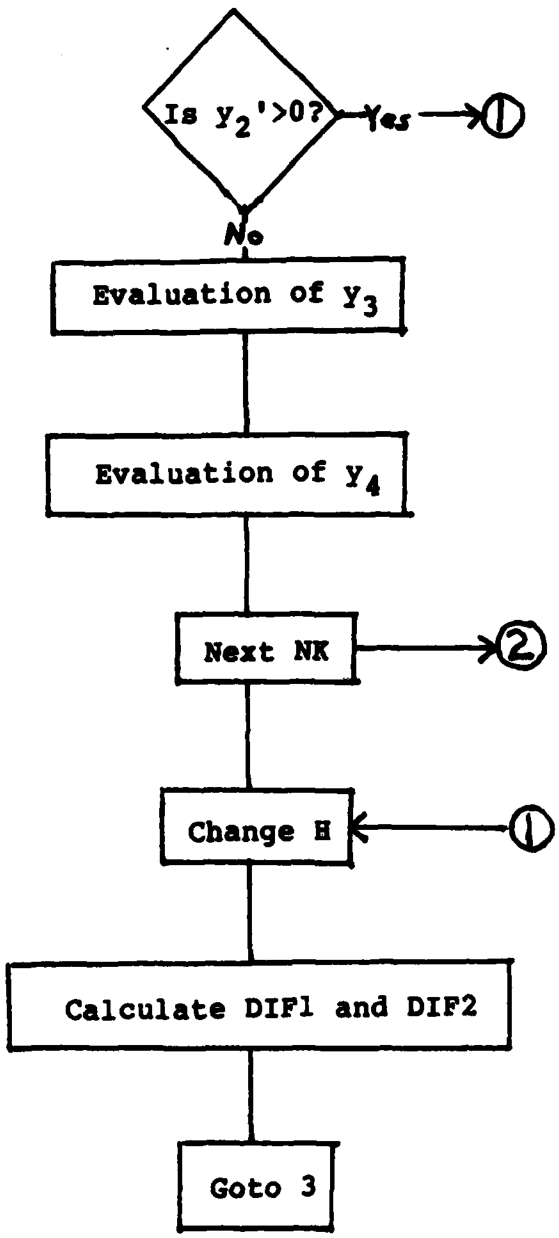
Y13#-43# -intermediate approximations for Y1#-4# contained within the integration routine

YPR1# -evaluation of Y1PRIME# for calculation of Y2#

YPRIME1#-4#-derivatives of Y1#-4# in integration routine

Z -value of z at which power series approximations are made. Z=T1


START
Calculation of constants
Evaluation of coefficients for power series expansions of y_1, y_2, y_3, y_4 using recursion relations C-11, C-14, C-16, C-23
Evaluation of y_1, y_2, y_3 and y_4 using power series expansions
3
Beginning of fourth order Runge-Kutta integration routine using values of y_1, y_2, y_3 and y_4 from power series approximations as initial values. Evaluation at mesh point NK.
2
Evaluation of y_1
Evaluation of y_2


Is y_2'>0?
Yes
1
No
Evaluation of y_3
Evaluation of y_4
Next NK
2
Change H
1
Calculate DIF1 and DIF2
Goto 3

```
1000 DIM R#(50),S#(50),P#(50),Q#(50)
1010 P=.4
1020 MNUN=4.8693E-16*P
1030 MNUP=8.8508E-16*P
1040 MNUE=2.9127E-18*P
1050 T0=.15
1060 TE=2.1
1070 SIGMA=T0/TE
1080 NO2=(P*1333!)/(8.3136*T0*11587*1.66E-24*100*100000!)
1090 C1=1.84E-11: C2=1.15: C3=47.7: C4=181: C5=251: C6=159:
     C7=110
1100 C8=52.6: C9=91.1: C10=64.2: C11=47.9: A=2.68E-09:
     B=4.61E-09
1110 C=.045: A1=.278393: A2=.230389: A3=.000972: A4=.078108
1120 DEF FNERF(X)=1-1/(1+A1*X+A2*X^2+A3*X^3+A4*X^4)^4
1130 ALPHA=NO2*C1*EXP(-3.8/TE)/TE^(3/2)*(.8862*EXP((C2/TE)^2)
           *(1-FNERF(C2/TE))*(C3 - C4/TE + C5/TE^2 - C6/TE^3
           + C7/TE^4) + C8 - C9/TE + C10/TE^2 - C11/TE^3)
1140 EI=12.063
1150 BETA=NO2*A*EXP(-EI*(1/TE - C))/(TE^(3/2))
          *1/(1/TE - C)*(1/(1/TE - C) + EI)-B*EXP(-EI/TE)
          *(TE^(.5) + EI*1/TE^.5)
1160 PRINT "alpha=";ALPHA,"beta=";BETA
1170 AA=MNUP*ALPHA/(TE*1.6E-12)
1180 BB=AA + MNUN*ALPHA/(TE*1.6E-12)
1190 CC=MNUE*ALPHA/(TE*1.6E-12)
1200 GAMMA=.03*NO2*1.1E-09
1210 PRINT "aa=";AA,"bb=";BB,"cc=";CC
1220 HI=100
1230 REM **EVALUATION OF COEFFICIENTS FOR POWER SERIES
           EXPANSIONS OF Y1, Y2, Y3 AND Y4**
1240 H=20
1250 L%=-1
1260 P#(0)=1
1270 Q#(0)=H
1280 SUM1#=0: SUM2#=0: SUM3#=0: SUM4#=0
1290 FOR I=0 TO 15
1300   R#(2*I+1)=(((BETA-ALPHA)/ALPHA)*P#(2*I) + (GAMMA/ALPHA)
                *Q#(2*I))/(2*(I+1))
1310   S#(2*I+1)=(P#(2*I) - (GAMMA/ALPHA)*Q#(2*I))/(2*(I+1))
1320   FOR J=1 TO I
1330     SUM1#=SUM1# + S#(2*J-1)*P#(2*(I+1-J))
1340     SUM2#=SUM2# + 2*J*Q#(2*J)*P#(2*(I+1-J))
1350     SUM3#=SUM3# + 2*J*P#(2*J)*Q#(2*(I+1-J))
1360     SUM4#=SUM4# + R#(2*J-1)*Q#(2*(I+1-J))
1370   NEXT J
1380   P#(2*I+2)=(((BB-AA)*(S#(2*I+1)+SUM1#) + SIGMA*SUM2#
                - SUM3# - CC*(H*R#(2*I+1)+SUM4#))/(2*(I+1))
                - (AA*R#(2*I+1) + BB*S#(2*I+1))/(4*(I+1)))
                /(.5+H)
```

```
1390   Q#(2*I+2)=(-(AA*R#(2*I+1) + BB*S#(2*I+1))/(2*(I+1))
                 - P#(2*I+2))/(2*SIGMA)
1400   L%=L%+1
1410 NEXT I
1420 Y1#=P#(0)
1430 Y2#=Q#(0)
1440 Y3#=0
1450 Y4#=0
1460 Z=.01
1470 REM **EVALUATION OF INITIAL VALUES OF Y1, Y2, Y3 AND Y4
           USING POWER SERIES EXPANSIONS**
1480 FOR K=1 TO L%+1
1490   Y1#=Y1#+P#(2*K)*Z^(2*K)
1500   Y1D#=Y1D#+P#(2*K)*2*K*Z^(2*K-1)
1510   Y2#=Y2#+Q#(2*K)*Z^(2*K)
1520   Y2D#=Y2D#+Q#(2*K)*2*K*Z^(2*K-1)
1530   Y3#=Y3#+R#(2*K-1)*Z^(2*K-1)
1540   Y3D#=Y3D#+R#(2*K-1)*(2*K-1)*Z^(2*K-2)
1550   Y4#=Y4#+S#(2*K-1)*Z^(2*K-1)
1560   Y4D#=Y4D#+S#(2*K-1)*(2*K-1)*Z^(2*K-2)
1570 NEXT K
1580 REM **RUNGA-KUTTA FOURTH ORDER METHOD**
1590 T1=.01
1600 TN=.99
1610 NN=100
1620 DT#=(TN-T1)/NN
1630 FOR NK=1 TO NN
1640   Y1T#=Y1#: Y2T#=Y2#: Y3T#=Y3#: Y4T#=Y4#
1650   T4#=T1+NK*DT#
1660   T3#=T4#-DT#/2
1670   T2#=T4#-DT#
1680   REM *Evaluation of Y1*
1690   Y10#=Y1#
1700   K1#=((BB-AA)*Y1#*Y4#-CC*Y2#*Y3#-Y10#*(AA*Y3#+BB*Y4#)/2)
           /(Y2#+Y1#/2)
1710   Y13#=Y1#+K1#*DT#/2
1720   K2#=((BB-AA)*Y13#*Y4#-CC*Y2#*Y3#-Y13#*(AA*Y3#+BB*Y4#)/2)
           /(Y2#+Y13#/2)
1730   Y13#=Y1#+K2#*DT#/2
1740   K3#=((BB-AA)*Y13#*Y4#-CC*Y2#*Y3#-Y13#*(AA*Y3#+BB*Y4#)/2)
           /(Y2#+Y13#/2)
1750   Y13#=Y1#+K3#*DT#
1760   K4#=((BB-AA)*Y13#*Y4#-CC*Y2#*Y3#-Y13#*(AA*Y3#+BB*Y4#)/2)
           /(Y2#+Y13#/2)
1770   Y1#=Y1#+(K1#+2*K2#+2*K3#+K4#)*DT#/6
1780   YPRIME1#=(K1#+K2#*2+2*K3#+K4#)/6
1790   REM *Evaluation of Y2*
1800   Y20#=Y2#
1810   YPR1#=((BB-AA)*Y10#*Y4#-CC*Y2#*Y3#-Y10#
            *(AA*Y3#+BB*Y4#)/2)/(Y2#+Y10#/2)
```

```
1820    K1#=(-AA*Y3#-BB*Y4#-YPR1#)/(2*.15/2.1)
1830    Y23#=Y2#+K1#*DT#/2
1840    YPR1#=((BB-AA)*Y10#*Y4#-CC*Y23#*Y3#-Y10#
              *(AA*Y3#+BB*Y4#)/2)/(Y23#+Y10#/2)
1850    K2#=(-AA*Y3#-BB*Y4#-YPR1#)/(2*.15/2.1)
1860    Y23#=Y2#+K2#*DT#/2
1870    YPR1#=((BB-AA)*Y10#*Y4#-CC*Y23#*Y3#-Y10#
              *(AA*Y3#+BB*Y4#)/2)/(Y23#+Y10#/2)
1880    K3#=(-AA*Y3#-BB*Y4#-YPR1#)/(2*.15/2.1)
1890    Y23#=Y2#+K3#*DT#
1900    YPR1#=((BB-AA)*Y10#*Y4#-CC*Y23#*Y3#-Y10#
              *(AA*Y3#+BB*Y4#)/2)/(Y23#+Y10#/2)
1910    K4#=(-AA*Y3#-BB*Y4#-YPR1#)/(2*.15/2.1)
1920    Y2#=Y2#+(K1#+2*K2#+2*K3#+K4#)*DT#/6
1930    YPRIME2#=(K1#+K2#*2+2*K3#+K4#)/6
1940    IF YPRIME2#>0 THEN GOTO 2220
1950    REM *Evaluation of Y3*
1960    Y30#=Y3#
1970    K1#=((BETA-ALPHA)/ALPHA)*Y10#+Y20#*GAMMA/ALPHA-Y3#/T2#
1980    Y33#=Y3#+K1#*DT#/2
1990    K2#=((BETA-ALPHA)/ALPHA)*Y10#+Y20#*GAMMA/ALPHA-Y33#/T3#
2000    Y33#=Y3#+K2#*DT#/2
2010    K3#=((BETA-ALPHA)/ALPHA)*Y10#+Y20#*GAMMA/ALPHA-Y33#/T3#
2020    Y33#=Y3#+K3#*DT#
2030    K4#=((BETA-ALPHA)/ALPHA)*Y10#+Y20#*GAMMA/ALPHA-Y33#/T4#
2040    Y3#=Y3#+(K1#+2*K2#+2*K3#+K4#)*DT#/6
2050    YPRIME3#=(K1#+K2#*2+2*K3#+K4#)/6
2060    REM *Evaluation of Y4*
2070    K1#=Y10#-Y20#*GAMMA/ALPHA - Y4#/T2#
2080    Y43#=Y4#+K1#*DT#/2
2090    K2#=Y10#-Y20#*GAMMA/ALPHA - Y43#/T3#
2100    Y43#=Y4#+K2#*DT#/2
2110    K3#=Y10#-Y20#*GAMMA/ALPHA - Y43#/T3#
2120    Y43#=Y4#+K3#*DT#
2130    K4#=Y10#-Y20#*GAMMA/ALPHA - Y43#/T4#
2140    Y4#=Y4#+(K1#+2*K2#+2*K3#+K4#)*DT#/6
2150    YPRIME4#=(K1#+K2#*2+2*K3#+K4#)/6
2160    EE=-ALPHA*Y3#*MNUE/(1.6E-19*Y1#*1E+07)
           - 1.6E-19*TE*YPRIME1#/(Y1#*1.6E-19)
2170    EN=-ALPHA*Y4#*MNUN/(1.6E-19*Y2#*1E+07)
           - 1.6E-19*T0*YPRIME2#/(Y2#*1.6E-19)
2180    PRINT T4#,EE,EN
2190    PRINT Y1#,Y2#/H,Y3#,Y4#
2200 PRINT
2210 NEXT NK
2220 H=H-.001
2230 LPRINT "H=";H+.001,"NK=";NK
2240 EP2=.01
2250 AA1=Y1#/EP2
2260 BB1=Y2#/EP2
```

```
2270 DD=AA1*AA*(BB-AA)+BB*CC*BB1
2280 CC1=(AA1*(BB-AA)*(AA1+2*SIGMA*BB1)+AA1*BB1*BB*(1-SIGMA))/DD
2290 DD1=-(AA*AA1*BB1*(1-SIGMA)-CC*BB1*(AA1+2*SIGMA*BB1))/DD
2300 DIF1=Y3#-CC1*(1+EP2)
2310 DIF2=Y4#-DD1*(1+EP2)
2320 LPRINT "dif1=";DIF1,"dif2=";DIF2
2330 GOTO 1250
```

APPENDIX E

Explanation of Thompson's Code

List of Variables

ALPHA -dissociative attachment rate (sec^{-1})

C2 -constant given by eq. B-54

DIF -difference between the left and right sides of eq. 3-6

DT -mesh interval

E -electric field (V/cm)

GAMMA - θ_e / θ_g

HI,LO -boundaries on x used in the iterative subroutine for finding x

K1-4 -estimate of first derivative of w in integration routine

L1-4 -estimate of second derivative of w in integration routine

M -coefficient of x^{γ} in definition of w ($M = \frac{2\theta_g h}{\theta_e + \theta_g}$)

MUE,MUI,

MUN -mobilities μ_e , μ_+ and μ_-

N -number of mesh intervals

NC -coefficient of x in definition of w (NC=1)

NU -ionization rate (sec^{-1})

Q -final estimate of the first detivative of w at the end of each iteration in the inegration routine

Q1 -initial value of Q

R -ratio of

SE,SI,SN -constants defined to be the three terms in the numerator of B-54

T1 -initial value of spatial coordinate throughout a given mesh interval

T2-4 -intermediate values of the spatial coordinate throughout a given mesh interval

THETAE, THETAG - θ_e and θ_g

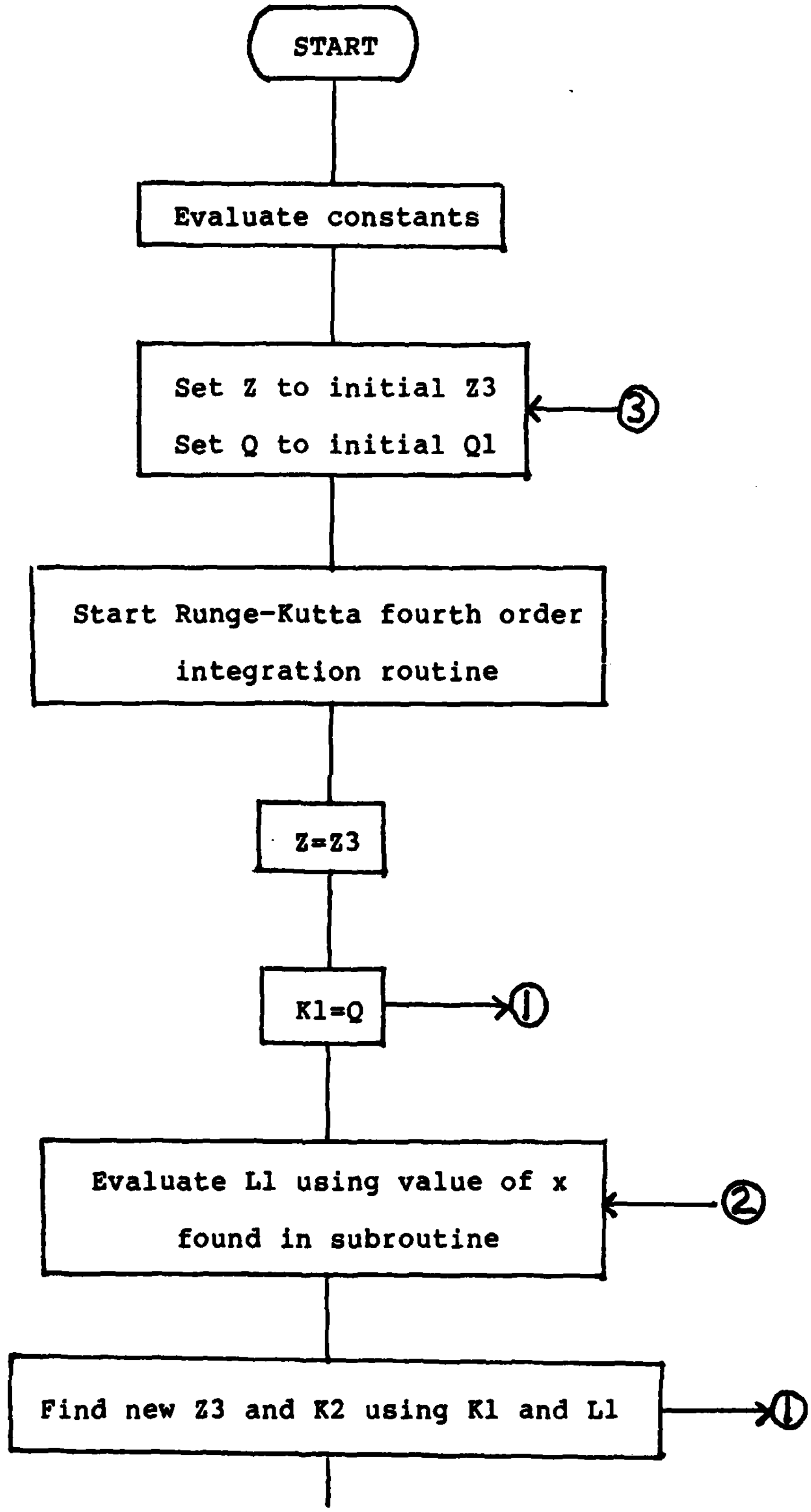
TN -final value of spatial coordinate where numerical integration ends

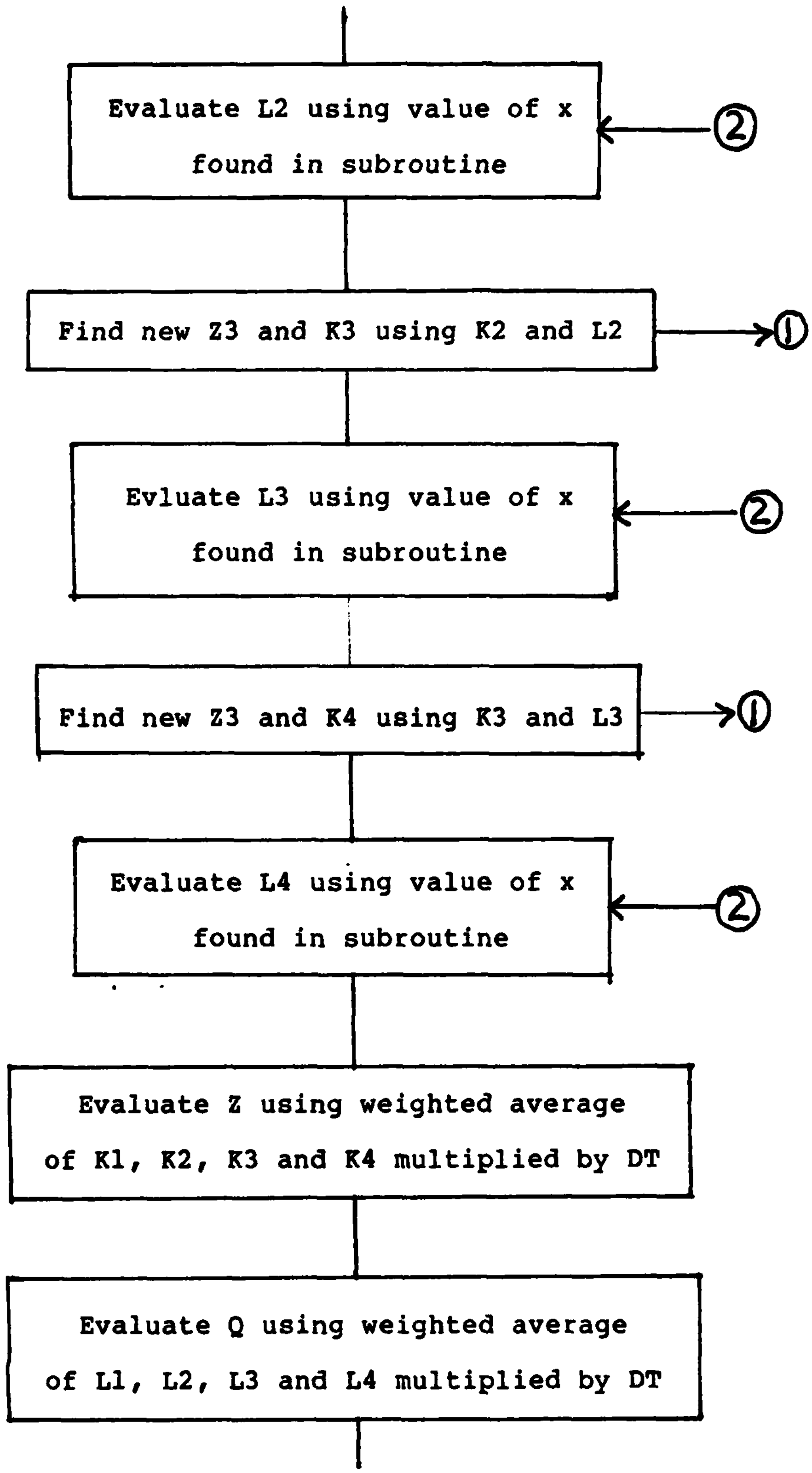
X -normalized electron density, $x = \frac{n_e}{n_{eo}}$

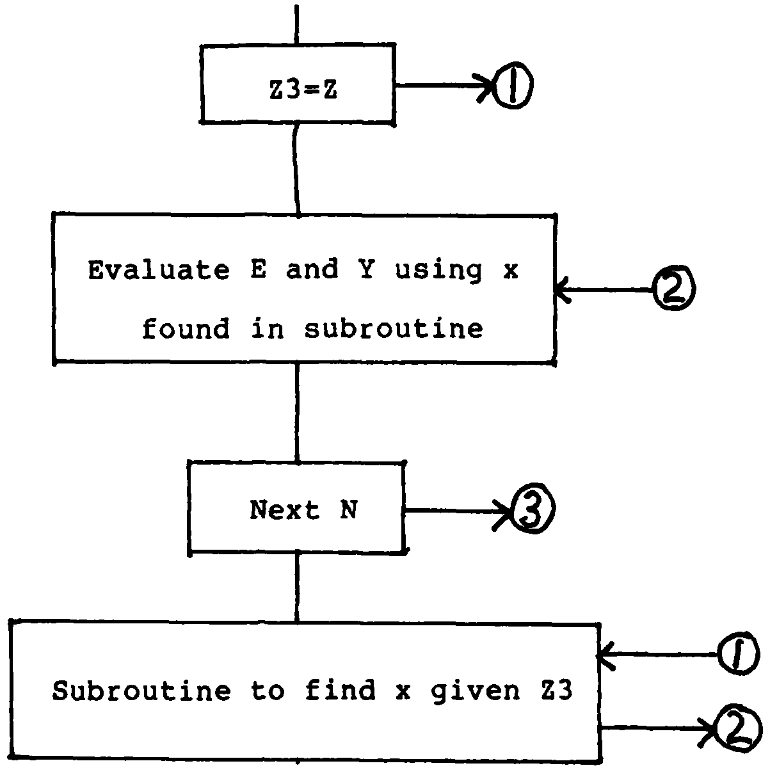
Y -normalized negative ion density, $y = \frac{n_-}{n_{-o}}$

Z -final estimate of w (as defined in 3-6) at the end of each iteration in the integration routine

Z3 -estimate of w in integration routine


START
Evaluate constants
Set Z to initial Z3
Set Q to initial Q1
3
Start Runge-Kutta fourth order
integration routine
Z=Z3
K1=Q
1
Evaluate L1 using value of x
found in subroutine
2
Find new Z3 and K2 using K1 and L1
1


Evaluate L2 using value of x found in subroutine
2
Find new Z3 and K3 using K2 and L2
1
Evluate L3 using value of x found in subroutine
2
Find new Z3 and K4 using K3 and L3
1
Evaluate L4 using value of x found in subroutine
2
Evaluate Z using weighted average of K1, K2, K3 and K4 multiplied by DT
Evaluate Q using weighted average of L1, L2, L3 and L4 multiplied by DT



To include associative detachment in the code for Thompson, the following lines must be included:

```
1045 RECOM=(insert associative detachment rate in sec^-1)
1046 D2=(RECOM/MUE - RECOM/MUN)/(THETAE + THETAG)
1525 Y=X^GAMMA
```

The following lines must be changed to:

```
1230 L1=-C2*X - D2*Y
1270 L2=-C2*X - D2*Y
1310 L3=-C2*X - D2*Y
1480 IF ABS(DIF)<.0001 THEN GOTO 1525
```

```
1000 NU=14.1: ALPHA=7
1010 MUE=1022.7: MUI=2.25: MUN=4.4
1020 SE=(NU-ALPHA)/MUE: SI=NU/MUI: SN=ALPHA/MUN
1030 THETAE=2.4: THETAG=.15
1040 C2=(SE+SI+SN)/(THETAE+THETAG)
1050 R=7
1060 M=2*THETAG*R/(THETAE+THETAG): NC=1
1070 GAMMA=THETAE/THETAG
1080 N=100
1090 T1=0
1100 TN=1
1110 Z3=M+NC
1120 Z=Z3
1130 Q1=0
1140 Q=Q1
1150 DT=(TN-T1)/N
1160 FOR N=1 TO 100
1170   Z3=Z
1180   T4=T1+N*DT
1190   T3=T4-DT/2
1200   T2=T4-DT
1210   K1=Q
1220   GOSUB 1430
1230   L1=-C2*X
1240   Z3=Z+K1*DT/2
1250   K2=Q+L1*DT/2
1260   GOSUB 1430
1270   L2=-C2*X
1280   Z3=Z+K2*DT/2
1290   K3=Q+L2*DT/2
1300   GOSUB 1430
1310   L3=-C2*X
1320   Z3=Z+K3*DT
1330   K4=Q+L3*DT
1340   GOSUB 1430
1350   L4=-C2*X
1360   Z=Z+(K1+2*K2+2*K3+K4)*DT/6
1370   Q=Q+(L1+2*L2+2*L3+L4)*DT/6
1380   Z3=Z
1390   GOSUB 1430
1400   E=MUP*THETAG*(1+GAMMA+2*ALPHA*GAMMA)*Q/(MUE*(1+MUP
         *(1+ALPHA)/MUE+ALPHA*MUN/MUE)*X) - THETAE*Q/X
1410   PRINT "t=";T4,"e=";E,"X=";X,"Y=";X^GAMMA
1420 NEXT N
1430 REM *SUBROUTINE--FIND X^GAMMA*
1440 LO=0
1450 HI=1
1460 X=.5
1470 DIF=Z3-M*X^GAMMA-NC*X
1480 IF ABS(DIF)<.0001 THEN GOTO 1530
```

```
1490 IF DIF<0 THEN HI=X
1500 IF DIF>0 THEN LO=X
1510 X=(LO+HI)/2
1520 GOTO 1470
1530 RETURN
```

APPENDIX F

Explanation of Ingold's Code

The symbols appearing in the following code have the same definitions given in appendix E with the exception of the following minor changes and additions:

AA,BB,
CC,DD -A,B,C and D as defined in B-51 and B-53
NN,MM,S -n,m and s as defined in B-49 and B-53

u replaces x and s replaces γ in the definitions of HI, LO, M and NC where the last two are now given by 3-2.

The flow chart is also similar to the one given in appendix E where the changes noted above must be made.

```
10 THETAE=2.4: THETAG=.15
20 GAMMA=THETAE/THETAG
30 MUE=1022.7: MUI=2.25: MUN=4.4
40 R=10
50 NU=14
60 ALPHA=NU*MUN*R/(MUE+MUN*R)
70 SE=(NU-ALPHA)/MUE: SI=NU/MUI: SN=ALPHA/MUN
80 A=GAMMA*(1+1/GAMMA)*SE
90 B=SI + SN - SE/GAMMA
100 C=SI - SN +SE
110 D=2*SE*R
120 F=(B-C-((B-C)^2 + 4*A*D)^.5)/(2*D)
130 PRINT F
140 NN=C+D*F: MM=B-D*F
150 S=NN/MM
160 PRINT S
170 BB=D/(MM-NN)
180 AA=(1-BB*(1-F))/(1-F)^S
190 CC=AA*F: DD=1+BB*F
200 PRINT "A=";AA,"B=";BB,"C=";CC,"D=";DD
210 C2=(SE+SI+SN)/(THETAE+THETAG)
220 M=AA+2*R*CC/(GAMMA+1): NC=BB+2*R*DD/(GAMMA+1)
230 N=100
240 T1=0
250 TN=1
260 Z3=M*(1-F)^S + NC*(1-F)
270 Z=Z3
280 Q1=0
290 Q=Q1
300 DT=(TN-T1)/N
310 FOR N=1 TO 100
320   Z3=Z
330   T4=T1+N*DT
340   T3=T4-DT/2
350   T2=T4-DT
360   K1=Q
370   GOSUB 570
380   L1=-C2*(AA*U^S + BB*U)
390   Z3=Z+K1*DT/2
400   K2=Q+L1*DT/2
410   GOSUB 570
420   L2=-C2*(AA*U^S + BB*U)
430   Z3=Z+K2*DT/2
440   K3=Q+L2*DT/2
450   GOSUB 570
460   L3=-C2*(AA*U^S + BB*U)
470   Z3=Z+K3*DT
480   K4=Q+L3*DT
490   GOSUB 570
500   L4=-C2*(AA*U^S + BB*U)
```

```
510   Z=Z+(K1+2*K2+2*K3+K4)*DT/6
520   Q=Q+(L1+2*L2+2*L3+L4)*DT/6
530   Z3=Z
540   GOSUB 570
550   PRINT "t=";T4,"X=";AA*U^S+BB*U,"Y=";CC*U^S+DD*U,"U=";U
560 NEXT N
570 REM *SUBROUTINE--FIND U*
580 LO=0
590 HI=1-F
600 U=.5
610 DIF=Z3-M*U^S -NC*U
620 IF ABS(DIF)<.0001 THEN GOTO 670
630 IF DIF<0 THEN HI=U
640 IF DIF>0 THEN LO=U
650 U=(LO+HI)/2
660 GOTO 610
670 RETURN
```

REPORT DOCUMENTATION PAGE

1a. REPORT SECURITY CLASSIFICATION
UNCLASSIFIED

1b. RESTRICTIVE MARKINGS

2a. SECURITY CLASSIFICATION AUTHORITY

2b. DECLASSIFICATION/DOWNGRADING SCHEDULE

3. DISTRIBUTION/AVAILABILITY OF REPORT
Approved for public release;
distribution unlimited.

4. PERFORMING ORGANIZATION REPORT NUMBER(S)
AFIT/GNE/~~ENG~~ ENP/85M-3

5. MONITORING ORGANIZATION REPORT NUMBER(S)

6a. NAME OF PERFORMING ORGANIZATION
School of Engineering

6b. OFFICE SYMBOL *(If applicable)*
AFIT/EN~~G~~P

7a. NAME OF MONITORING ORGANIZATION

6c. ADDRESS *(City, State and ZIP Code)*
Air Force Institute of Technology
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8a. NAME OF FUNDING/SPONSORING ORGANIZATION

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9. PROCUREMENT INSTRUMENT IDENTIFICATION NUMBER

8c. ADDRESS *(City, State and ZIP Code)*

10. SOURCE OF FUNDING NOS.

PROGRAM ELEMENT NO.	PROJECT NO.	TASK NO.	WORK UNIT NO.

11. TITLE *(Include Security Classification)*
See Box 19

12. PERSONAL AUTHOR(S)
Christopher J. Clouse, B.S., 2d Lt, USAF

13a. TYPE OF REPORT	13b. TIME COVERED	14. DATE OF REPORT (Yr., Mo., Day)	15. PAGE COUNT
MS Thesis	FROM ______ TO ______	1985 January	112

16. SUPPLEMENTARY NOTATION

17. COSATI CODES

FIELD	GROUP	SUB. GR.
20	09	

18. SUBJECT TERMS *(Continue on reverse if necessary and identify by block number)*
Three Component Diffusion, Electronegative Gases,
Oxygen Discharge Tube,

19. ABSTRACT *(Continue on reverse if necessary and identify by block number)*

Title: SCHOTTKY THEORY OF THREE COMPONENT PLASMAS

Thesis Chairman: William Bailey, Lt Col, USAF

Approved for public release: IAW AFR 190-1.
18 april 85
[illegible]
Dean for [illegible] Professional Development
[illegible] Technology (AIC)
Wright-[illegible]

20. DISTRIBUTION/AVAILABILITY OF ABSTRACT
UNCLASSIFIED/UNLIMITED ☒ SAME AS RPT. ☐ DTIC USERS ☐

21. ABSTRACT SECURITY CLASSIFICATION
Unclassified

22a. NAME OF RESPONSIBLE INDIVIDUAL
William Bailey, Lt Col, USAF

22b. TELEPHONE NUMBER *(Include Area Code)*

22c. OFFICE SYMBOL
AFIT/ENG

The purpose of this study was to analyze and compare papers ~~by Lee (ref. 7), Thompson (refs. 9 and 10) and Ingold (ref. 6)~~ which give conflicting results concerning the charged particle profiles in an oxygen discharge tube. Lee predicts proportional negative ion and electron profiles whereas Thompson and Ingold predict non-proportional profiles.

The analytic developments were critically reviewed and numerical solutions were developed for each approach. It was found that Thompson's results could not be obtained without introducing the additional assumption:

$$\frac{\nabla n_-}{\nabla n_e} = \gamma\alpha$$

Ingold's development seemed to be lacking a firm mathematical basis. One of Ingold's fundamental relations used throughout his development could not be established and, furthermore, it was shown that a completely different solution could be developed without intrducing additional constraints.

Lee's profiles reduce to Thompson's when associative detachment is ignored and a common set of parameters and coordinate system are used. When associative detachment is included, Lee and Thompson do not give similar profiles. Thus, Thompson's relation, as stated above, may only be valid in limiting cases (e.g. when associative detachment can be ignored.) Results from an investigation of power series solutions indicated that they could probably be used to obtain Lee's profiles, this negating the need to numerically integrate simultaneous first order equations, as is done by Lee.

cont keywords include: sec. 1-73

END

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